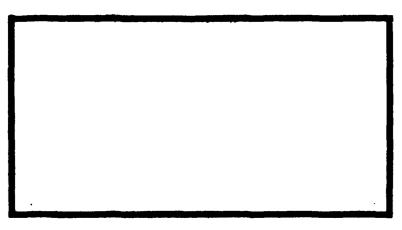AD-A115 496   AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH   SCHOO--ETC   F/G 12/1
                A NEW GOODNESS OF FIT TEST FOR NORMALITY WITH MEAN AND VARIANCE--ETC(U)
                DEC 81 T J REAM
UNCLASSIFIED   AFIT/GOR/MA/81D-9   NL

1 OF 2
AD A
115496

AD A115496

UNITED STATES AIR FORCE
AIR UNIVERSITY
AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

DTIC
SELECTE**D**
JUN 1 4 1982

E

82 06 14 183

DTIC FILE COPY

A NEW GOODNESS OF FIT TEST FOR
NORMALITY WITH MEAN AND VARIANCE
UNKNOWN

THESIS

Thomas J. Ream
AFIT/GOR/MA/81D-9   Capt   USAF

DTIC
ELECTE
JUN 1 4 1982

E

AFIT/GOR/MA/81D-9

A NEW GOODNESS OF FIT TEST FOR NORMALITY

WITH MEAN AND VARIANCE UNKNOWN

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

Requirements for the Degree of

Master of Science

by

Thomas J. Ream
Capt   USAF

Graduate Operations Research

December 1981

Approved for public release; distribution unlimited.

Acknowledgments

Contents

List of Figures

List of Tables

Abstract

A new technique for calculating known goodness of fit statistics for the Normal distribution is investigated. Samples are generated for a Normal (0,1) distribution. The means of these samples are calculated and the samples are doubled by reflecting sample data points about the individual sample means. This reflection of data points about the mean is the new technique for generating modified statistics. After the sample is doubled, critical values are calculated for these modified Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises statistics. Critical values are for the original sample sizes. An extensive power study is done to test the power of the three new statistics' critical values versus the power for the same three statistics, calculated without reflection.

Powers of the new statistics are asymptotically slightly higher than the powers of their non-reflected counterparts, when the distribution tested is also symmetrical. The powers of new statistics are substantially lower when the distribution tested is non-symmetrical. The powers are substantially higher for the modified statistics when the continuous Uniform distribution is tested.

Complete tables of critical values for sample sizes $n = 3$ through $n = 60$ are included for the modified Kolmogorov-Smirnov and Anderson-Darling statistics.

A NEW GOODNESS OF FIT TEST FOR NORMALITY
WITH MEAN AND VARIANCE UNKNOWN

I. Introduction

This thesis is an investigation of a technique that
involves doubling samples by reflecting the sample data
points about their arithmetic mean before calculating good-
ness of fit statistics. Tables are to be generated for the
Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises
statistics using this technique. The usefulness of the
tables is demonstrated by a comprehensive power study.

General Comments About Goodness of Fit

Goodness of Fit--Definition. Goodness of fit is
based on the idea that one can take a set of data and deter-
mine how well it corresponds (or fits) with some known dis-
tribution. "Goodness" refers to the quality of this fit.

Typical Non-parametric Test. In the area of non-
parametric statistics, most goodness of fit procedures attempt
to establish a statistical test of fit which relies on a yes/
no decision rather than some measure of "goodness." The
typical test uses a null hypothesis, H_0: the data are from
some known continuous distribution. The alternative, H_A, is
that the data are not from the hypothesized distribution.
Typically, the analyst is hoping to accept H_0. The purpose

1

of these tests is to determine if the data are distributed similarly enough to the hypothesized distribution to ascribe the properties of the hypothesized distribution to the population from which the data were taken. For example, if the analyst has a group of data he thinks is distributed exponentially, he could use one of the goodness of fit tests to reach a statistical conclusion about whether or not the population from which the data are drawn is exponential.

For the more common theoretical distributions, tables of critical values have been derived for different goodness of fit statistics. One of these tables has been derived by Lilliefors for the Kolmogorov-Smirnov (K-S) statistic and the normal distribution with the parameters estimated from the sample (Lilliefors, 1967). To use his tables, one would calculate the statistic and compare it with the critical value. If the calculated statistic were greater than the critical value for the desired α-level (α is the probability that H_0 is rejected when H_0 is true), H_0 (that the data being tested are normally distributed) would be rejected (Lilliefors, 1967).

Power Problems. Since the non-parametric test involves a yes/no decision rather than some proportional measure of goodness, the power of a given statistic is very important to the analyst. Power is the probability of rejecting H_0 when H_A is, in fact, true (Mendenhall & Schaeffer, 1973). The power of a given test provides some measure of the quality of the statistical test itself. Thus, the power

2

is a measure of the degree of usefulness of the goodness of fit test. If the power is low, then one cannot state the distribution of the data with as much confidence as if the power had been high.

One of the problems with many of the goodness of fit statistics is that, with smaller sample sizes (n = 10), they are not very powerful. (Throughout this paper, the term "powerful" will be used to mean "of or having relatively high power.") This lack of power is evident for the normal distribution, in particular, even against skewed distributions (Green & Hegazy, 1976; Stephens, 1974). None of the statistics, for which Green and Hegazy reported powers, had powers greater than 0.5 when sample size was ten (Green & Hegazy, 1976).

Another problem with goodness of fit tests is that they are more powerful against some distributions than they are against others (Lilliefors, 1967; Stephens, 1974; Green & Hezagy, 1976). In that sense, power study results are again useful to the analyst. For example, suppose H_0 is that some sample of data is drawn from a normal population. Suppose the calculated goodness of fit test statistic is .087. Suppose the critical value for that statistic is .079. The test statistic value is greater than the critical value, so H_0 would be rejected. In that case, the analyst could refer to a power study and perhaps find that for this particular statistic, the power versus the exponential is .97. He could then state with high confidence that the data is

3

not exponential, but normal. From another power study, he might also find that the power versus the double exponential is .36. Thus, he could not have as much confidence in a statement that the data are not from a double exponential, but from a normal population.

This research effort is an investigation of a new method that will, hopefully, provide more powerful goodness of fit tests for three of the common goodness of fit statistics. The new method is the doubling of samples about the sample mean before calculating the statistic. This technique is applied to calculating critical values for the normal distribution.

Three Test Statistics. The three test statistics being used have been tested for their power when calculated for the normal distribution (Stephens, 1974; Green & Hegazy, 1976; Lilliefors, 1967). These previous tests suggest a methodology for the power studies done using the technique being investigated here. The statistics which will be used are the Kolmogorov-Smirnov (K-S) statistic (Massey, 1951), the Anderson-Darling (A-D) statistic (Anderson & Darling, 1954), and the Cramer-von Mises (CVM) statistic (Anderson & Darling, 1954).

The statistics are discussed in greater detail in Chapter II, the background chapter of this report. It is important to note that all statistics in this research are calculated after estimating the mean and variance from the sample data.

4

Primary Research Issue

The new statistical technique studied in this research is motivated by the work of Schuster (1973; 1975). Schuster suggests that samples of symmetrical distributions can be reflected about the parameter of symmetry to generate a new sample with identical parameters. He uses this concept to develop a new statistic that uses two samples, the original one and the reflected one (Schuster, 1973). The technique suggested by this author results in a different statistic than Schuster's. However, the statistics probably are not totally dissimilar. Both Schuster's and this author's techniques can be expected to have similar characteristics because they both use reflection.

The New Technique. The logic of the technique proposed by the author follows. If a sample of some size, n (e.g., n = 10), is taken from a normal population, the actual number of points used to calculate the test statistic can be doubled about the arithmetic mean of the sample data. In other words, rather than calculating the critical values for the normal distribution at n = 10 with ten data points, twenty actual points will be used. The technique is demonstrated with an example in Chapter II.

More Restrictive Critical Values. It is felt that the use of this reflection technique will result in the generation of more restrictive critical values. Because the critical values supposedly will be more restrictive, it is possible that the probability of rejecting H_0 when H_A is

true will increase. In other words, the possibility that the power will be greater when data points are reflected about their mean will be investigated.

General Research Hypothesis. The general hypothesis being used to guide this research can be stated as follows:

Hypothesis: For the normal distribution, the K-S, A-D and CVM statistics, modified by calculation after doubling the sample by reflecting data points about the sample mean, provide more powerful tests of goodness of fit than do the same statistics calculated without reflection.

While it is hypothesized that generally more powerful statistics will result from reflection, some implications from Schuster's work should be considered since he also used reflection. First, Schuster proved that for his statistic better results could only be expected when alternative distributions are also symmetric (Schuster, 1973). One would, thus, not be surprised to find higher powers only versus symmetrical distributions for the new statistic. Second, Schuster only obtained better results asymptotically when the parameters were estimated from the sample. In other words, his statistic was "better" only for larger sample sizes (Schuster, 1973). It should not be surprising if this is also the case for the new technique.

Primary Purpose. The primary purpose of this thesis is to test the above hypothesis and to generate tables of critical values for the three previously mentioned statistics,

modified by doubling the sample by reflection. While the basic hypothesis being tested is presented in the previous paragraph, several other techniques are to be tested before developing the computer programs for generation of critical values. These are briefly described in the following paragraphs. More detailed discussions are presented in Chapter II.

Bootstrap Technique

Continuous vs. Discrete. Prior critical value tables have been determined by calculating and ordering statistics for a large number of random samples from the test distribution. If 1000 statistics are calculated, the critical value for α = .05 is the 950th largest order statistic. The process uses discrete values to determine critical values for continuous distributions.

The bootstrap technique developed by Efron (1979) and recently demonstrated by Johnston (1980) is a method for representing these order statistics on a continuous spectrum. This is done by plotting the values of the order statistics and representing the spaces between them as piecewise linear functions (Efron, 1979; Johnston, 1980).

Interpolation. If the order statistics are plotted versus a plotting position that would represent each of the order statistics on a scale between zero and one, it is possible to interpolate for the desired percentile and, therefore, extract a more accurate value. It is also possible

7

that by using this technique, cost savings can be realized, since fewer random deviates may have to be generated in order to get consistent critical values at the desired α levels.

Plotting Positions

As mentioned above, the bootstrap technique requires the use of some plotting position to scale the order statistics between zero and one. Three different plotting positions are tested to see if there is any noticeable arithmetic difference among them with large numbers ($n > 100$). The three plotting positions tested are called the median rank, a modified step rank, and the average of mean and mode ranks. These three plotting positions are presented in detail in Chapter II.

If the differences among the three plotting positions are judged to be minor, only one of the positions will be used. If there are major differences, then critical values will be calculated using all three positions, and only the most powerful results will be tabled.

The reason these positions are the ones being tested is that they all have a desired symmetrical property. They all provide symmetry in the following sense. Suppose one has a graph with order statistics on the horizontal axis and plotting position on the vertical axis. The vertical component of the plot at the first order statistic is identical to the quantity: one minus the vertical component at the last order statistic.

Presentation of Research

The report on this thesis effort is presented in five chapters. The first of these is this introduction. Although the introduction is meant to be detailed enough for a reader familiar with the research area, Chapter II is a background chapter for the use of anyone interested in more details about the techniques that have been discussed in the introduction.

The methods used to examine the above techniques are presented in Chapter III. The results of the research described in Chapter III are presented in Chapter IV. Chapter IV is a discussion of what happened. Tables of critical values and results of power studies are located in this chapter. The final chapter consists of conclusions and recommendations.

Primary Purpose Reemphasized

The primary purpose of this research effort is to test the technique of reflecting data points about the mean and to create tables of critical values of the modified K-S, A-D, and CVM statistics for the normal distribution using that technique. Statistics are calculated using normalized data with the mean and variance estimated from the sample.

II. Background

In the previous chapter, the basic concepts and techniques being studied in this thesis were presented. This chapter explains some of those techniques in greater detail. The chapter is divided into five sections. These include some introductory comments; a presentation of the three plotting positions to be examined; a discussion of the K-S, A-D, and CVM statistics; further explanation of the bootstrap technique; an example of doubling samples by reflecting them about their means; and a summary.

Introductory Comments

Purpose. The purpose of this chapter is to present more detailed discussions of some of the techniques mentioned in Chapter I. This chapter is meant to be used as a reference chapter. One familiar with the research area might not need to read this chapter.

Format. The format is different than that used in Chapter I. The sequence is now the order in which the ideas are studied in the research. The following is a list of the topics in the order of discussion:

    a. Plotting positions

    b. Three statistics

        1. K-S (Kolmogorov-Smirnov)
        2. A-D (Anderson-Darling)
        3. CVM (Cramer-von Mises)

      c.   Bootstrap technique

      d.   Doubling samples about the mean

Plotting Positions

Why? From Chapter I, the reason the plotting posi-
tion is necessary is to provide a vertical plot scaled between
zero and one. A vertical plot is required for each value of
the order statistic represented on the horizontal axis.
Consider drawing n samples and calculating the same statistic
for each sample. The results would be a set of n statistics.
When ordered, the set is of n order statistics. Given the
set of order statistics, $X_{(1)}$, $X_{(2)}$, $X_{(3)}$, ..., $X_{(n)}$, n is
the total number of statistics and i is the rank of a given
statistic, i = 1,2,3,...,n. For example, the rank of $X_{(3)}$
is 3, or i = 3. Letting the value of order statistics be
represented by the horizontal axis and letting the vertical
axis be scaled between zero and one, the plotting positions
being tested allow the statistics to represent points on a
continuous function.

For example, let n = 10 samples. Suppose this re-
sulted in the ten statistic values in order $(X_{(i)})$ listed
below. If one used the median rank (which is defined later)
as the vertical plotting position $(Y_{(i)})$, he would get the
list as shown on the next page. These values are plotted in
Fig. 1. If straight lines are drawn between the plotted
positions, a piecewise linear continuous function results.

In the research, each of the three plotting positions

11
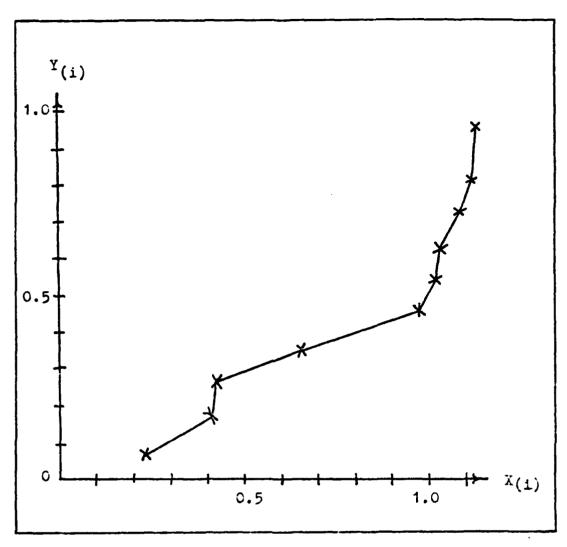
Fig. 1.   Example (Order Statistics vs Median Ranks)

i	$X_{(i)}$	Median Rank ($Y_{(i)}$)
1	.22	.067
2	.41	.163
3	.42	.260
4	.67	.356
5	.98	.452
6	1.02	.548
7	1.03	.644
8	1.08	.740
9	1.12	.837
10	1.13	.933

will be examined to see if there is much difference among
them with values of n greater than 99.

The plotting positions are now described in detail.

Median Rank. The formula for median rank is as
follows:

$$\text{median rank} = \frac{i - 0.3}{n + 0.4} \qquad (1)$$

where

i = rank of order statistic being plotted

n = total number of order statistics

The above formula is well known. From the example, suppose
the statistic being plotted is $X_{(5)}$ = 0.98. In this case,
where n = 10, the median rank is as follows:

$$\text{median rank} = \frac{5 - 0.3}{10 + .4} = \frac{4.7}{10.4} = .452 \qquad (2)$$

A property of this plotting position worth noting
is that $X_{(1)}$ is the same distance from zero as $X_{(n)}$ is from
one. For instance at n = 10, the median rank for $X_{(1)}$ =
.067 and for $X_{(10)}$ = .933. Let the median rank of $X_{(i)}$ be
defined as $Y_{(i)}$. Then, $Y_{(1)}$ - 0.0 = .067 and 1.0 - $Y_{(10)}$ =
.067. This is the desired symmetry discussed in Chapter I.

Modified Step Rank. The second ranking procedure
discussed is the modified step rank. To understand this,
one must first know the step rank formula. The formula for
the step rank is also well known and is as follows:

$$\text{step rank} = \frac{i - 1}{n} \qquad (3)$$

The reason this formula needs to be modified is that it does not have the same type of symmetry as that shown for the median rank. For example, again let n = 10, then for i = 1, the step rank is 0.0. For i = 10, the step rank is 0.9. If $Y_{(i)}$ is the step rank of $X_{(i)}$, then $1 - Y_{(10)} = 0.1$ and $Y_{(1)} - 0.0 = 0.0$. The desired symmetry does not exist.

The desired symmetry can be obtained if the following modification is made:

$$\text{modified step rank} = \frac{i - 0.5}{n} \qquad (4)$$

Let $Y_{(i)}$ be the modified step rank of $X_{(i)}$. Then, at n = 10, $Y_{(1)} = 0.5$ and $Y_{(10)} = 0.95$. If follows that $Y_{(1)} - 0 = 0.05$ and $1 - Y_{(10)} = 0.05$. Hence, the desired symmetry exists.

Average of Mode and Mean Ranks. The last plotting position discussed uses the average of the mode and mean ranks. The formulas for the mean and mode ranks are also well known. Three ranks are presented below--the mode rank, the mean rank, and the average of the two:

$$\text{mean rank} = \frac{1}{n + 1} \qquad (5)$$

$$\text{mode rank} = \frac{i - 1}{n - 1} \qquad (6)$$

$$\text{average} = \frac{\frac{i}{n+1} + \frac{i-1}{n-1}}{2} \qquad (7)$$

The mode and mean ranks do not have the desired symmetry about zero and one. The average of those two ranks does. Though not done here, this fact can be easily demonstrated.

14

Three Statistics

This section is a presentation of the three statistics being studied. All statistics will be discussed as they apply to the normal distribution. The parameters of the normal distribution, μ and σ, are unknown and will be estimated for each sample by their maximum likelihood estimators, $\bar{x}$ and S (Mendenhall & Scheaffer, 1973), where

$$x = \frac{\sum_{i=1}^{n} x_i}{n} \tag{7}$$

$$S^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \tag{8}$$

$$S = \sqrt{S^2} \tag{9}$$

Kolmogorov-Smirnov (K-S) Statistic.

The common symbol for the K-S statistic is D. The statistic is defined (Massey; 1951; Lilliefors, 1967) as

$$D = \max|F^*(x) - S_N(x)| \tag{10}$$

where

the sample data points are ordered,

$F^*(x)$ = normal CDF value of a given data point,

$S_N(x)$ = sample cumulative step function.

$\bar{x}$ and S are needed to find $F^*(x)$. $S_N(x)$ has two values for each ordered data point. These values are i/n and $(i-1)/n$, where i is the rank of the ith ordered data point and n is the sample size. The following is an example of how to

15

calculate the K-S statistic for a given sample:

x	$S_N(x)$		$F^*(x)$	$\lvert F^*(x) - S_N(x) \rvert$	
0.2	0.000	0.125	0.1038	.1038	.0212
1.6	.125	.250	.2033	.0783	.0467
2.1	.250	.375	.2483	.0017	.1267
3.0	.375	.500	.3446	.0304	.1554
4.8	.500	.625	.5596	.0596	.0654
5.0	.625	.750	.5871	.0379	.1629
8.1	.750	.875	.8790	.1290	.0040
9.6	.875	1.000	.9484	.0734	.0516

$$D = \max \lvert F^*(x) - S_N(x) \rvert = .1629$$

$$\bar{x} = 4.3, \quad S = 3.249, \quad n = 8$$

<u>Anderson-Darling (A-D) Statistic</u>. A common notation
for the A-D statistic is W^2 (Anderson & Darling, 1954). Let
$X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \cdots \leq X_{(n)}$ be n observations from the
sample in order. Let $u_i = F(X_{(i)})$ = the normal CDF value
with $\bar{x}$ and S as estimators of μ and σ. Then, the A-D statis-
tic (Anderson & Darling, 1954) is

$$W^2 = -n - \frac{1}{n} \sum_{j=1}^{n} (2j-1)[\ln u_j + \ln(1 - u_{n-j+1})] \quad (11)$$

Letting $A = \ln u_j$ and $B = \ln(1 - u_{n-j+1})$, the following is
a numerical example using the same data points as the K-S
sample:

j	x	$F(x) = u_j$	u_{n-j+1}	A	B	(2j-1)(A+B)
1	0.2	.1038	.9484	-2.265	-2.964	- 5.229
2	1.6	.2033	.8790	-1.593	-2.112	-11.115
3	2.1	.2483	.5871	-1.393	- .885	-11.390
4	3.0	.3446	.5596	-1.065	- .820	-13.195

16

j	x	F(x) = u_j	u_{n-j+1}	A	B	(2j-1)(A+B)
5	4.8	.5596	.3446	- .581	-.423	- 9.036
6	5.0	.5871	.2483	- .533	-.285	- 8.998
7	8.1	.8790	.2033	- .129	-.227	- 4.628
8	9.6	.9484	.1038	- .053	-.110	- 2.445

$$\Sigma = -66.036$$

$$A\text{-}D = W^2 = -8 - (1/8)(-66.036)$$

$$= -8 - 8.2545$$

$$= \underline{.2545}$$

<u>Cramer-von Mises (CVM) Statistic</u>. The Cramer-von Mises statistic (Anderson & Darling, 1954) is the third to be studied in this research.

Let n = sample size,

$u_i = F(X_{(i)})$ = CDF value for normal distribution,

and

$X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \cdots \leq X_{(n)}$ be n observations in order,

then

$$CVM = \frac{1}{12n} + \sum_{j=1}^{n} [u_j - \frac{(2j-1)}{2n}]^2 \qquad (12)$$

The following is a numerical example of calculation of the CVM statistic:

j	x	F(X) = u_j	A = (2j-1)/2n	$(u_j - A)^2$
1	0.2	.1038	.0625	.00171
2	1.6	.2033	.1875	.00025
3	2.1	.2483	.3125	.00412
4	3.0	.3446	.4375	.00863

17

j	x	F(X) = u_j	A = (2j-1)/2n	$(u_j - A)^2$
5	4.8	.5596	.5625	.00001
6	5.0	.5871	.6875	.01008
7	8.1	.8790	.8125	.00442
8	9.6	.9484	.9375	.00012

$$\Sigma = .02934$$

$$CVM = 1/(12)(8) + .02934$$

$$= .01042 + .02934$$

$$= \underline{.03976}$$

Bootstrap Technique

The bootstrap technique is used in this thesis as it was demonstrated by Johnston (1980). One of the three plotting positions tested will be used to represent the vertical axis from zero to one. The value of the n test statistics will be the horizontal components. Lines between the plots will be interpolated, as was demonstrated in Fig. 1.

Extrapolation. In addition to the interpolations, extrapolations are necessary to find values for $X_{(0)}$ and $X_{(n+1)}$, where $X_{(i)}$ is the ith order statistic, i = 0,1,2, ...,n,n+1. If $Y_{(i)}$ represents the vertical rank determined by one of the ranking procedures, $Y_{(1)}$ is greater than zero and $Y_{(n)}$ is less than one. Since a vertical scale from zero to one is desirable in order to find critical values for any level of significance between zero and one, values of $X_{(0)}$ and $X_{(n+1)}$ must be found for $Y_{(0)} = 0$ and $Y_{(n+1)} = 1$.

To find $X_{(0)}$, the slope of the line between $X_{(1)}$ and $X_{(2)}$ is determined. That line is then extrapolated to

its intercept with the x-axis. If the intercept is greater than or equal to zero, then $X_{(0)}$ equals the intercept value. If the intercept is less than zero, then $X_{(0)} = 0$. (Since all of the statistics being tested yield non-negative values, $X_{(0)}$ cannot be allowed to be negative.) The line between $X_{(0)}$ and $X_{(1)}$ is, then, interpolated.

To find $X_{(n+1)}$, the same technique is used, except negative values are not a problem. The line between $X_{(n-1)}$ and $X_{(n)}$ is formed. That line is then extended to its intercept with the line $Y_{(i)} = 1$. The intercept value is the value for $X_{(n+1)}$.

Figure 2 is a display of the above three situations. Graph (a) depicts the situation where the x-intercept is less than zero. In that case, the solid line is the line from $(X_{(0)}, Y_{(0)})$ to $(X_{(1)}, Y_{(1)})$. Graph (b) is the case in which the x-intercept is greater than or equal to zero. Graph (c) of Fig. 2 represents finding $X_{(n+1)}$.

Finding the Critical Value. To find a critical value, all that is necessary, graphically, is to find $1 - \alpha$ on the vertical axis and extend along the line, $Y_{(i)} = 1 - \alpha$, to intercept the plotted function. The value of the horizontal component is the critical value of the statistic at significance level α.

Finding the critical value with a computer requires finding the largest $Y_{(i)}$ that is less than $1 - \alpha$. Suppose that $Y_{(i)}$ is the kth largest rank. Then, the standard linear slope-intercept formula ($y = mx + b$) is used to find the

19
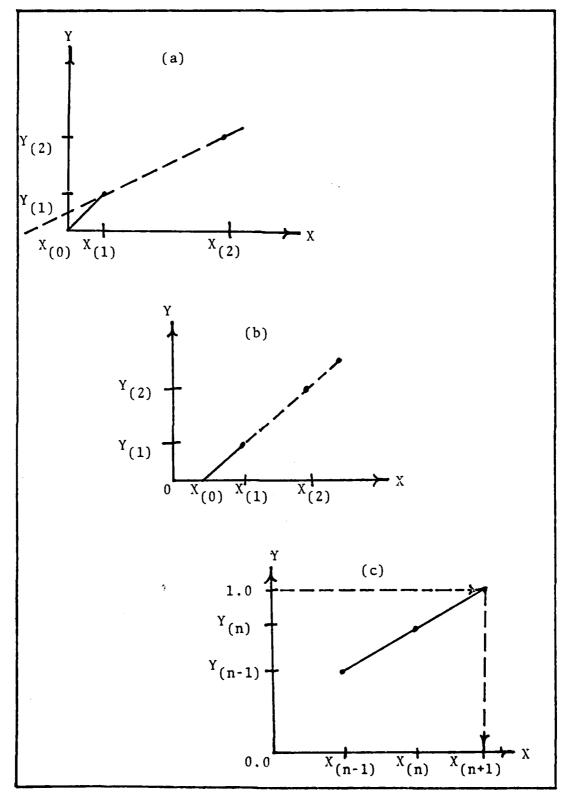
Fig 2. Three Examples of Extrapolation

20

critical value. The change in y can be found using $Y_{(k)}$ and $Y_{(k+1)}$. Similarly, the change in x can be found using $X_{(k)}$ and $X_{(k+1)}$. After finding the constant, b, at $(X_{(k)}, Y_{(k)})$, one can then let y equal 1 - α in order to find x, the critical value.

Example of Technique. As in the example in Fig. 1, suppose ten samples are taken. Let the following numbers be the ten statistics calculated:

i	Modified Step Rank $(Y_{(i)})$	Statistics $(X_{(i)})$
1	.05	.22
2	.15	.41
3	.25	.42
4	.35	.67
5	.45	.98
6	.55	1.02
7	.65	1.03
8	.75	1.08
9	.85	1.12
10	.95	1.13

In Fig. 3, the statistics are plotted versus their modified step ranks. From the above list,

$$Y_{(1)} = 0.05$$
$$Y_{(2)} = 0.15$$
$$X_{(1)} = 0.22$$
$$X_{(2)} = 0.41$$

Using the equation, $y = mx + b$,

$$m = \text{slope} = \frac{Y_{(2)} - Y_{(1)}}{X_{(2)} - X_{(1)}} = \frac{.15 - .05}{.41 - .22} = 0.52$$

21
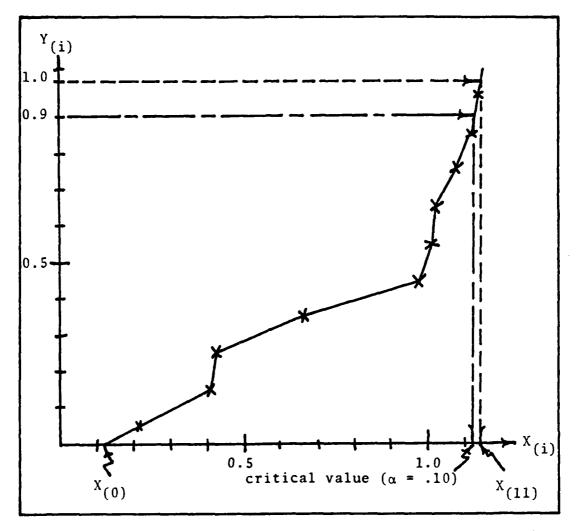
Fig 3. Example of Bootstrap Technique

$$b = Y_{(1)} - mX_{(1)}$$

$$b = .05 - (.52)(.22) = .065$$

and

$$x = \frac{0.0 - b}{m} = \frac{.065}{.52} = .125$$

Since x = .125 $\geq$ 0, $X_{(0)}$ = x. Again, if x had been less than

zero, $X_{(0)}$ would have been set equal to zero.

Extrapolation for $X_{(11)}$ is performed the same way.

22

$Y_{(10)}$, $Y_{(9)}$, $X_{(10)}$, and $X_{(9)}$ are used to find the slope. The constant, b, is calculated at either $(X_{(10)}, Y_{(10)})$ or at $(X_{(9)}, Y_{(9)})$. Then, $X_{(11)} = [(1.0-b)/m]$, where m is the slope.

Now that the function is continuous (by extrapolation) on the interval (0,1), the critical values can be found. At $\alpha = .10$, previous studies (Lilliefors, 1967; Green & Hegazy, 1976; Anderson & Darling, 1954; Massey, 1951) would have picked 1.12, or the ninth largest statistic as the critical value. Using the bootstrap method, the value is 1.125 (if modified step ranks are used).

To get the critical value using the bootstrap technique, the largest $Y_{(i)}$ less than or equal to .90 is found. In this case, this is $Y_{(9)} = .85$. Therefore, k = 9 and k + 1 = 10. Then,

$$m = \frac{Y_{(10)} - Y_{(9)}}{X_{(10)} - X_{(9)}} = \frac{.95 - .85}{1.13 - 1.12} = \frac{.10}{.01} = 10$$

$$b = .85 - (10)(1.12) = .85 - 11.2 = -10.35$$

and

$$\text{critical value} = \frac{.90 - (-10.35)}{10} = 1.125$$

As one can see, the critical value will vary with statistics calculated for random samples. One of the issues of this research is the number of samples needed to get consistent results.

Doubling Samples About the Mean

The following is a description of the technique of doubling samples about their means. First, a sample of
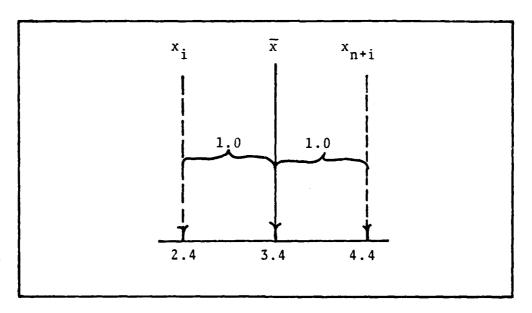
Fig 4.   Doubling Samples About the Mean

random deviates is collected.   Second, the arithmetic mean
is calculated.

The third step has several sub-steps.   Let i = 1,2,
3,...,n, and n be the number of random deviates in a sample.
Then, the new deviate (created by reflection about the mean)
is $x_{n+i} = 2\bar{x} - x_i$.   Looking at Fig. 4, suppose $x_i = 2.4$
and the mean of all the x_i's is $\bar{x} = 3.4$.   Then, x_{n+i}
$= 2(3.4) - 2.4 = 4.4$.   Notice that both points are equidis-
tant from the mean.   The mean from the newly created sample
is the same as the original one.

Example.   An example is presented in Table I.   The
first column is the five data points in the original sample.
The second column is of the left-hand sides of five equations,
representing $2\bar{x} - x_i$ for each data point.   The third
column is the reflected data point.

24

TABLE I

Reflection of Data Points About the Mean

Data Points (n = 5)	$2\bar{x} - x_i$		Reflected Data Point
0.2	2(3.4) - 0.2	=	6.6
1.6	2(3.4) - 1.6	=	5.2
2.1	2(3.4) - 2.1	=	4.7
5.0	2(3.4) - 5.0	=	1.8
8.1	2(3.4) - 8.1	=	-1.3

Before reflection: $\bar{x} = 3.4$

After reflection: $\bar{x} = 3.4$

Summary

This chapter is a set of detailed discussions of techniques referred to in Chapter I. The techniques discussed are plotting positions (ranking techniques), the three statistics studied, the bootstrap technique, and the procedure of doubling samples about their arithmetic mean. Specific references will be made to this chapter in the following chapter on procedure.

III. Procedure

The techniques to be used in the experimental pro-
cedure have been presented in detail in Chapters I and II.
This chapter is a discussion of how those techniques are to
be applied. Since all of the data are generated using
Monte Carlo simulation of pseudo-random deviates, this is
essentially a chapter about how the previously discussed
techniques are combined into computer programs to generate
and manipulate Monte Carlo data for testing the research
hypothesis presented in Chapter I.

This chapter has four major sections. The first is
about how the three plotting positions are to be tested.
The second concerns the calculation of statistics and their
critical values. The third section is a discussion of the
generation of tables of critical values. In the last sec-
tion, the construction of the power study is presented.

Plotting Positions

The purpose of the first phase of research is to
compare three plotting positions. The search is for meaning-
ful differences among the median rank (M), modified step rank
(MS), and the average of the mean and mode ranks (AMM) at
various values of n (n is the number of statistics to be
plotted). If there are meaningful arithmetic differences,
all three will be used. If no meaningful differences exist,

only the modified step will be used for simplicity. "Meaning-ful" is an intentionally loose term. The researcher cannot judge whether the differences are important, or "meaningful", until he has seen what the differences actually are. The plotting positions, themselves, are thoroughly discussed in Chapter II.

The Computer Program. Since visual comparisons of plotting positions for each value of i (i = 1,2,3,...,n) are desired, the comparison is done via computer. The program used is simple and is included in Appendix A. The program has the following three major steps:

1. For some n, find the value for each plotting position at every i, i = 1,2,3,...,n

2. Find the differences among the three plotting positions at each value of i.

3. Print out for every value of i:

a. the values of the three positions

b. the absolute value of:

1. M - MS

2. M - AMM

3. AMM - MS

The program is run for n = 100, n = 150, and n = 300 statistics.

Calculation of Statistics and Critical Values

The calculation of statistics for random samples is at the heart of this research effort. All programs that are used calculate statistics. All either calculate or use

previously calculated critical values. The point is that all the programs use much of the same flow and code to accomplish these calculations.

The four basic steps used in the programs include the following:

1. Calculating statistics (using different sub-routines for each statistic),

2. Storing the statistics in a vector array,

3. Ordering the elements of that array from smallest to largest, and

4. Calculating critical values using the bootstrap method that was discussed in Chapter II.

Subprogram for Calculating Statistics. The logic for that portion of each program that deals with calculating the statistic is shown in Fig. 5. The letter on the right-hand side of each block is the block identifier.

Subprogram for Finding Critical Values. The logic for that portion of each program that is used to find the critical values is shown in Fig. 6.

Testing the Program. The program can be tested for validity, since tables of critical values for the straight-forward calculation of the Kolmogorow-Smirnov (K-S) statistic are readily available. With 5000 samples, the program can be run without estimating the parameters, i.e., assuming $\mu = 0$ and $\sigma = 1$. These critical values can be compared with those obtained by Massey (Massey, 1951, p. 70). Once this is done, the program is modified as shown in Fig. 7. The

```
┌──────────────────────────────────────────────────────────┐
│                                                            │
│        ┌─────────────────────────────────┬─────┐          │
│        │     Generate ordered random     │     │          │
│        │     deviates from Normal (0,1)   │  A  │          │
│        └─────────────────────────────────┴─────┘          │
│                          ↓                                 │
│        ┌─────────────────────────────────┬─────┐          │
│        │         Find CDF value for       │     │          │
│        │          each data point         │  B  │          │
│        └─────────────────────────────────┴─────┘          │
│                          ↓                                 │
│        ┌─────────────────────────────────┬─────┐          │
│        │          Calculate K-S           │     │          │
│        │            statistic             │  C  │          │
│        └─────────────────────────────────┴─────┘          │
│                          ↓                                 │
│        ┌─────────────────────────────────┬─────┐          │
│        │       Store statistic in a       │     │          │
│        │     vector array of length n     │  D  │          │
│        └─────────────────────────────────┴─────┘          │
│                                                            │
│                  Reiterate the above                       │
│                     flow n times                           │
│                                                            │
└──────────────────────────────────────────────────────────┘
```

Fig 5.   Subprogram for Calculating Statistics

```
┌──────────────────────────────────────────────────────────┐
│                                                            │
│        ┌─────────────────────────────────┬─────┐          │
│        │          Order the Array         │     │          │
│        │           of statistics          │  A  │          │
│        └─────────────────────────────────┴─────┘          │
│                          ↓                                 │
│        ┌─────────────────────────────────┬─────┐          │
│        │     Extrapolate for the 0th      │     │          │
│        │   and n+1st order statistics     │  B  │          │
│        └─────────────────────────────────┴─────┘          │
│                          ↓                                 │
│        ┌─────────────────────────────────┬─────┐          │
│        │         Find critical value      │     │          │
│        │          using bootstrap         │  C  │          │
│        └─────────────────────────────────┴─────┘          │
│                                                            │
└──────────────────────────────────────────────────────────┘
```

Fig 6.   Subprogram for Finding Critical Values

```
┌─────────────────────────────────────────────────────────┐
│                                                           │
│                   ┌──────────────────┐                    │
│                   │   Block A of      │                   │
│                   │   Figure 5        │                   │
│                   └──────────────────┘                    │
│                            ↓                              │
│          ┌─────────────────────────────┬──────┐           │
│          │   Calculate x̄ and S          │ A-1  │           │
│          │      for sample             │      │           │
│          └─────────────────────────────┴──────┘           │
│                            ↓                              │
│          ┌─────────────────────────────┬──────┐           │
│          │        Calculate            │      │           │
│          │                             │      │           │
│          │           xᵢ - x̄            │ A-2  │           │
│          │      zᵢ = ─────             │      │           │
│          │             S               │      │           │
│          │                             │      │           │
│          │  for each data point (xᵢ)   │      │           │
│          └─────────────────────────────┴──────┘           │
│                            ↓                              │
│          ┌─────────────────────────────┬──────┐           │
│          │   Replace original data     │ A-3  │           │
│          │    point (xᵢ) with zᵢ        │      │           │
│          └─────────────────────────────┴──────┘           │
│                            ↓                              │
│                   ┌──────────────────┐                    │
│                   │   Block B of      │                   │
│                   │   Figure 5        │                   │
│                   └──────────────────┘                    │
│                                                           │
└─────────────────────────────────────────────────────────┘
```

The block A-2 contains:

$$z_i = \frac{x_i - \bar{x}}{S}$$

for each data point (x_i)

Fig 7.   Program Logic for Standardizing
the Data

three logic blocks in Fig. 7 fit between blocks A and B of

Fig. 5.   With this modification, the program will generate

critical values after estimating the parameters of the nor-

mal by $\bar{x}$ and S and standardizing the data.   When this is done

with 5000 samples, the results can be compared with those of

Lilliefors (Lilliefors, 1967, p. 400).

The Number of Samples to Use. The next research issue to investigate is if the bootstrap method will allow the use of considerably less than 5000 order statistics to calculate the critical values.

To do this test, critical values are calculated for the K-S statistic using 150, 300, 500, 1000, and 5000 samples. All samples are generated by Monte Carlo simulation and using different seeds. If the values are essentially the same at 500, 1000, and 5000 samples, then critical values for tables can be calculated using only 500 samples. Similarly, if the values are the same for 300, 500, 1000, and 5000, then 300 samples would be enough. The point is that if the researcher wants to use 300 samples to generate tables of critical values, the critical values at 300 must be the same as those calculated using 500, 1000, and 5000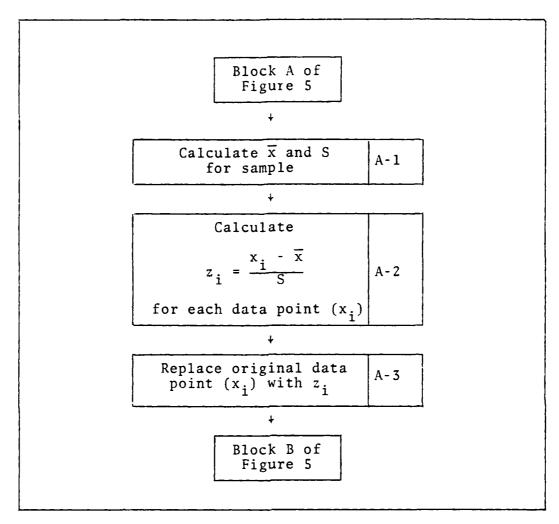. Five thousand samples is the number of samples commonly used in the literature to generate tables. The hope is that fewer will be needed by using the bootstrap technique. Whatever number of samples are used, however, must be consistent with the results at 5000 samples to be acceptable.

In addition to this vertical comparison, cross comparison with critical values found using different initial seeds to the random number generator are necessary. In one vertical comparison, the values might be essentially the same at 500, 1000, and 5000 samples. However, using a different seed, this may not hold true. The only consistency might be at 1000 and 5000. The critical values must be consistent

31

for a given number of samples--no matter what seed is used--
if that number of samples is going to be used to construct
valid, accurate tables.

The program used to test this issue is included in
Appendix D.

Program Subunits (Author's). Several subroutines
have been written by the author for use in the various pro-
grams. The code for these subroutines is included in Ap-
pendix C. The purposes and names of these subroutines are
discussed in the following paragraphs.

Three subroutines are used in the calculation of the
K-S statistics. These are CVALS, LILDIF, and DSTAT. ANDAR
is used to calculate the Anderson-Darling statistics, while
CVM is used to calculate the Cramer-von Mises statistics.

In addition to the five above, four subroutines are
used in a variety of programs. Their names and uses are
listed below:

ESTPAR - Takes an input vector array of data points
(x_i) and calculates $\bar{x}$ and S. It then
standardizes the data via the transforma-
tion,

$$z_i = \frac{x_i - \bar{x}}{S}$$

and outputs a vector array of standardized
data points (z_i).

DUBSAM - Takes an input vector array of length n,
calculates the mean of the vector elements,

reflects vector elements about the mean, and generates an output array of length 2n, which includes the original array elements plus their reflections.

XPOLAT - Used as part of the bootstrap technique. Input is a vector array of no order statistics. It extrapolates for $X_{(n+1)}$ and $X_{(0)}$. Output is an array of length, n + 2.

CVALUE - Input is an array of order statistics. Output is a set of critical values based on the elements of that array.

Program Subunits (IMSL). In addition to the author's own subroutines, several subroutines from the International Mathematical and Statistics Library (IMSL) are used. These include the following:

GGNO - Generates an array of ordered $N(0,1)$ random deviates.

MDNOR - For an input data point, outputs the CDF value of the standard normal distribution.

VSRTA - Orders the elements of an input array from smallest to largest.

Generation of Tables of Critical Values

Once the number of samples needed to get accurate critical values has been determined, the next step in the research is the generation of critical value tables. The tables to be generated are for the Kolmogorov-Smirnov (K-S),

33

Anderson-Darling (A-D), and Cramer-von Mises (CVM) statistics for sample sizes n = 3 through n = 60. The critical values are for modified statistics--statistics calculated after sample data points are reflected about the sample mean.

At this point the researcher is faced with a choice. The choice is between using a complex program that produces an entire table of critical values or using a simple program and reiterating it for each sample size. The second option is chosen despite the fact that it forces manual construction of the tables. This disadvantage is outweighed by the much more rapid computer turnaround for the simple program.

As a result of using the simpler methodology, each final table requires the submission of 171 programs-- 57 for K-S, 57 for A-D, and 57 for CVM. These individual programs are similar to the one described in the previous sections of this chapter. The only change is that in these programs, the samples are doubled by reflection about the sample means. This is done by subroutine DUBSAM after generating the random deviates and before standardizing the data. This step occurs between logic block A of Fig. 5 and logic block A-1 of Fig. 7. The program will generate critical values for α = .20, .15, .10, .05, and .01 for a given value of n.

In addition to the above programs, twelve more are required to generate critical values for use in the power study. Since the powers are to be compared at n = 10, 25, 40, and 60, critical values at α = .20, .15, .10, .05, and .01 must be determined without reflecting the sample. This

34

is done for each of the three statistics at each of the above four values of n.

Power Study

The purpose of the power study is to test the research hypothesis that the technique of reflecting data points about their means will result in goodness of fit tests with higher powers than ones which do not use that technique.

The power study is done at n = 10, n = 25, n = 40, and n = 60. The reasons for using these sample sizes are that 1) power comparisons will be available for both small and large sample sizes, and 2) trends in the behavior of the statistics' critical values can be observed.

The logic of the power study program follows. First, a sample is drawn from some distribution other than the normal. Second, the test statistic is calculated. Third, a comparison is made between the test statistic and the critical value for each level of α. If the test statistic is greater than the critical value, normality is rejected. The first three steps are then reiterated 5000 times. Each rejection is counted. The power at each α-level is computed by dividing the number of rejections by 5000. The results are then printed out.

Six statistics are calculated for each value of n. These statistics are the following:

1. K-S
2. K-S reflected

3. A-D

4. A-D reflected

5. CVM

6. CVM reflected

As with the generation of tables, the choice is made
here to submit simple programs and then construct tables
manually. Thus, to find the power of the statistics for the
normal against some other distribution, four programs are
required--one for each value of n. So, if seeking the power
against five distributions, twenty programs are required.
Different seeds are used for each run.

Flow of Typical Program. Figure 8 is a display of
the logic of the typical program used in the power study.
The flow in Fig. 8 is for finding the power of each of the
six statistics against the exponential distribution at sample
size, n = 10. The code for this particular program is in-
cluded in Appendix F as an example of the FORTRAN code used.

The Distributions Used. The distributions used in
this power study are the exponential, Cauchy, chi-squared
with four degrees of freedom, the chi-squared with one degree
of freedom, and the double exponential. The exponential
random deviates are generated by the IMSL subroutine, GGEXN.
The Cauchy deviates are generated by GGCAY (IMSL), and the
chi-squared ones are generated by GGCHS (IMSL).

The IMSL does not include a subroutine for the double
exponential. Therefore, double exponential deviates are
generated using the following technique. Continuous uniform

36

Generate sample of 10 exponential random deviates	A
Calculate the six statistics	B
Compare with their corresponding critical values (reject if test statistic > critical value)	C
Count rejections at each α- level	D
Reiterate steps A through D 5000 times	E
Calculate the powers at each α- level	F
Print the number of rejections and the powers	G
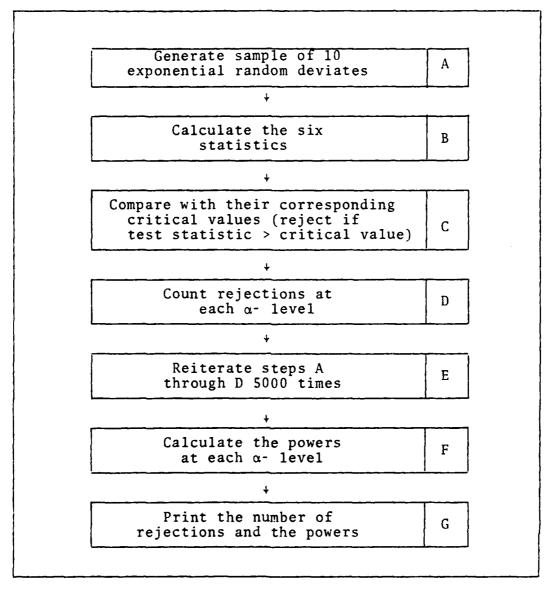
Fig 8.   Flow for Typical Power Study Program

random deviates, U_i, are generated by GGUBS (IMSL).   The CDF of the double exponential $[F(y_i)]$ is as follows:

$$F(y_i) = \begin{cases} \frac{1}{2}e^{y_i} , & y_i \leq 0 \\ 1 - \frac{1}{2}e^{-y_i} , & y_i > 0 \end{cases}$$

37

Therefore, if $U_i \leq 0.5$, then $y_i = \ln(2U_i)$, and

if $U_i > 0.5$, then $y_i = -\ln(2 - 2U_i)$.

Thus, y_i, $i = 1,2,\ldots n$, is a pseudo-random sample from the double exponential distribution (Littel, McClave, and Offen, 1979, p. 265).

Programs in the Appendices

An example of each type of program described in this chapter is included as an appendix. The following is a list of the appendices and the type of program or information included in each:

Appendix A: COMPAR - the program for comparing plotting positions.

Appendix B: Results of COMPAR - the results of program, COMPAR, when 150 points are to be plotted.

Appendix C: Subroutines - the computer code for the subroutines written by the author.

Appendix D: COMLIL - the program used to validate the logic used in finding critical values for the Kolmogorov-Smirnov statistic. This program is used to determine the number of samples to use for the bootstrap technique.

Appendix E: TABLE2 - the program for finding critical values of the modified Anderson-Darling statistic.

Appendix F: POWERS - the program for finding the critical values of the six statistics at sample size, n = 10, when the Cauchy is the alternative distribution.

All programs are written in FORTRAN V and are run on the Control Data Systems CDC 6600 computer which is operated by the Aeronautical Systems Division at Wright-Patterson AFB, Ohio.

Summary

This chapter is a presentation of the basic methodology used in the research. Flow diagrams are used to portray typical logic used in the different computer programs. The presentation includes discussions of 1) how plotting positions are compared, 2) how statistics and critical values are calculated, 3) how the tables of critical values are generated, and 4) how the power study was done.

The next chapter is a presentation of the results of this research.

IV. Results

This chapter is a presentation of the results of the research procedures described in the previous chapter. First to be discussed are the results of testing the three plotting positions. The section on plotting positions is followed by a section which reports the appropriate number of samples to use when finding the critical values. This is followed by the two major sections of the chapter--ones in which the tables of critical values and the results of the power study are presented. The chapter ends with a brief summary.

Test of Plotting Positions

The purpose of this testing of the plotting positions was to determine if there was any noticeable difference among the three. The results of the program using n = 150 (where n is the number of points to be plotted) are included in Appendix B.

With n = 150, the average of the mean and mode ranks (AMM) is essentially the same as the modified step rank (MS). The largest difference at n = 150 is 2.0×10^{-5}. At n = 300, the maximum difference is 1×10^{-5}.

In contrast, the differences between the median rank (M) and the other two is larger (by a factor of 10^2) at n = 150. The largest difference between AMM and M is 1.34×10^{-3}. The largest difference between MS and M is

1.32×10^{-3}. The difference is halved when n = 300.

Although the median rank is different than the other two plotting positions, the difference is still quite small. This difference becomes very, very small as the number of points to plot increases. Because the differences become slight as n increases, the decision was made to use the modified step rank in all calculations of critical values.

Test of the Program

As a test, the program for generating critical values was run with 5000 samples of sizes n = 10, n = 20, and n = 30. As stated in Chapter III, this was done for the Kolmogorov-Smirnov statistic so that the results could be compared with tables previously published.

The program which carried the assumption of normality, with $\mu = 0$ and $\sigma = 1$, generated critical values which were the same as Massey's (Massey, 1951). When the parameters of the normal distribution were estimated by $\bar{x}$ and S, the results were similar to those obtained by Lilliefors (Lilliefors, 1967). The program is, thus, valid.

The Number of Samples Used

The program for testing the consistency of critical values was run four times with a different seed each time. The program generated critical values using 150, 300, 500, 1000, and 5000 samples. The only number of samples that yielded consistent results through all four programs at all levels of α was 5000. If α = .01 had not been desired for

41

the critical value tables, 1000 samples appeared to generate critical values similar enough to each other to be used at the other levels of α. However, since $\alpha = .01$ was desired, 5000 samples were generated for each sample size from n = 3 to n = 60 for each statistic.

Tables of Critical Values

Only two complete tables of critical values are presented. Table II is a list of critical values for the Kolmogorov-Smirnov statistic when the sample is reflected about the mean. Table III is the same information for the modified Anderson-Darling statistic.

Only a partial table is presented for the Cramer-von Mises statistic. Table generation was stopped because the preliminary results of the power study were not promising for any of the statistics. Upon completion of the power study, it was found that the modified CVM statistic was rarely better than the modified A-D statistic. The decision was made to not waste computer resources generating a table of apparently minimal utility.

For the power study, however, critical values of the Cramer-von Mises statistic were needed for n = 10, n = 25, n = 40, and n = 60. A list of the critical values at these values of n is included as Table XV.

Use of the Tables. The following is the sequence of steps necessary to use Tables II, III, and XV.

    1. Collect data (sample size = n)

42

2. Double the sample by reflection about the mean (as described in Chapter III).

3. Standardize the data by the following transformation:

$$z_i = \frac{x_i - \bar{x}}{S}$$

where

x_i = the original data point

z_i = the standardized data point

$\bar{x}$ = the sample mean

S = the sample standard deviation

4. Calculate the statistic (see Chapter II).

5. Enter the table at the desired α-level and appropriate value of n.

6. If the statistic is greater than the table value, reject H_0: the data are from a normal population.

The tables are located on subsequent pages.

Power Study

The power study was initially done versus five continuous distributions. A power study computer program was also run using standard normal random deviates to validate the study. The following is a list of the distributions used and their corresponding tables:

1. Exponential (Table IV)
2. Cauchy (Table V)
3. Chi-squared with one degree of freedom (Table VI)
4. Chi-squared with four degrees of freedom (Table VII)
5. Double exponential (Table VIII)

43

Notes About the Tables. Several things should be noted about the tables. The first note is explanatory. The column headed "calculation method" has two symbols listed. The use of a single asterisk (*) indicates that the powers in that row are for straightforward calculation of the statistic. The use of a double asterisk (**) indicates that the powers in that row are for calculation of the statistic after doubling the sample by reflection about the arithmetic mean of the original sample.

The second item of note is that when the power is greater when the reflection technique is used versus when straightforward calculation is used, the power in the (**) row is underlined.

The third point is that if one peruses Tables IV through VIII, he will not find very many instances when the doubled asterisked power is underlined. When it is underlined, it is for a symmetric distribution. In the case of the Cauchy (Table V), one will notice: 1) that there is minimal power improvement and 2) that improvement is with large sample sizes. Most improvement is seen with the double exponential, although still only with relatively large sample sizes (Table VIII).

More Distributions. Because the improved power appeared to be against symmetrical unimodal distributions, it was decided to do additional power studies with the logistic and Student's t with three degrees of freedom. A study was done against the uniform just to see what would

44

TABLE II

Critical Values of the Modified Kolmogorov-Smirnov
Statistic for the Normal Distribution
(Parameters Estimated from the Sample)

n	.20	.15	α-level .10	.05	.01
3	.20144	.21133	.22155	.23119	.23832
4	.19040	.20749	.22981	.25958	.29949
5	.18600	.19626	.20955	.22566	.25164
6	.16981	.17746	.18786	.20432	.24933
7	.15883	.16973	.18258	.20179	.23340
8	.14923	.15861	.16924	.18591	.21416
9	.14279	.15057	.16106	.17807	.20894
10	.13452	.14278	.15309	.16858	.20295
11	.12990	.13734	.14731	.16308	.19580
12	.12535	.13225	.14163	.15788	.18677
13	.12046	.12708	.13649	.15045	.18021
14	.11654	.12344	.13171	.14519	.17046
15	.11272	.11924	.12818	.14087	.16812
16	.10893	.11483	.12334	.13621	.16301
17	.10721	.11287	.12029	.13293	.15758
18	.10334	.10929	.11811	.12912	.15234
19	.10152	.10724	.11447	.12630	.15069
20	.09938	.10500	.11187	.12217	.14629
21	.09732	.10321	.11096	.12284	.14452
22	.09416	.09965	.10693	.11735	.13980
23	.09337	.09849	.10548	.11523	.13596
24	.09005	.09543	.10246	.11350	.13664
25	.08818	.09328	.09931	.11045	.13294
26	.08777	.09302	.09986	.11062	.13309
27	.08608	.09120	.09780	.10760	.12760
28	.08498	.08957	.09583	.10612	.12750
29	.08254	.08753	.09404	.10381	.12427
30	.08144	.08635	.09190	.10019	.12038

TABLE II, continued

n	α-level				
	.20	.15	.10	.05	.01
31	.07965	.08393	.09045	.09957	.11916
32	.07892	.08361	.08945	.09986	.11821
33	.07734	.08170	.08806	.09756	.11564
34	.07769	.08144	.08750	.09657	.11538
35	.07582	.08003	.08570	.09352	.11001
36	.07436	.07874	.08381	.09204	.10847
37	.07389	.07808	.08367	.09170	.10814
38	.07355	.07781	.08314	.09134	.10898
39	.07137	.07534	.08076	.08934	.10689
40	.07103	.07530	.08069	.08895	.10428
41	.07001	.07409	.07905	.08811	.10513
42	.06954	.07352	.07928	.08722	.10372
43	.06838	.07225	.07712	.08524	.10186
44	.06768	.07160	.07741	.08539	.10215
45	.06721	.07137	.07680	.08435	.10097
46	.06683	.07025	.07503	.08331	.09976
47	.06623	.06991	.07443	.08182	.09830
48	.06511	.06904	.07465	.08186	.09744
49	.06374	.06781	.07295	.08038	.09496
50	.06363	.06690	.07226	.08051	.09377
51	.06347	.06766	.07239	.08056	.09575
52	.06253	.06614	.07085	.07849	.09352
53	.06205	.06534	.07006	.07751	.09103
54	.06153	.06488	.06965	.07704	.09189
55	.06111	.06477	.06941	.07682	.09342
56	.06070	.06433	.06902	.07631	.09068
57	.05938	.06299	.06739	.07477	.08903
58	.05995	.06304	.06773	.07561	.09026
59	.05923	.06249	.06703	.07393	.08664
60	.05828	.06166	.06608	.07301	.08669

TABLE III

Critical Values of the Modified Anderson-Darling Statistic for the Normal Distribution (Parameters Estimated from the Sample)

n	.20	.15	.10	.05	.01
3	.32197	.33708	.35211	.38220	.41027
4	.38954	.44834	.54657	.71621	1.01573
5	.41998	.47814	.55492	.65032	.74460
6	.40458	.44691	.50501	.62504	.95894
7	.41369	.46504	.54431	.67666	.94119
8	.42964	.47950	.55208	.69138	.94023
9	.43903	.49149	.70801	.70801	1.06640
10	.44203	.50275	.57780	.71245	1.05927
11	.44488	.50165	.59112	.73335	1.05248
12	.43843	.49746	.57663	.73101	1.08325
13	.04478	.49320	.57152	.72240	1.03260
14	.44722	.50810	.59214	.74576	1.07173
15	.45345	.51024	.59637	.74049	1.11475
16	.45242	.51498	.60171	.76214	1.14640
17	.46114	.51875	.60122	.76680	1.17257
18	.44973	.50160	.58093	.73484	1.12748
19	.44482	.51126	.59408	.75451	1.12149
20	.46305	.52665	.60590	.74583	1.05373
21	.45638	.51138	.59375	.75196	1.08071
22	.45134	.50675	.58571	.74983	1.15273
23	.46409	.53008	.62019	.76239	1.11597
24	.45368	.51731	.59381	.76619	1.07996
25	.45905	.52107	.61331	.76620	1.16808
26	.45657	.51742	.60062	.76192	1.14596
27	.46406	.52359	.61168	.75096	1.09041
28	.45768	.52019	.61730	.77880	1.17745
29	.45206	.51637	.60195	.74786	1.09862
30	.45293	.50558	.59092	.74476	1.10891

TABLE III, continued

n	α-level				
	.20	.15	.10	.05	.01
31	.45543	.51505	.60163	.74662	1.12114
32	.46263	.52015	.60162	.74479	1.13510
33	.46745	.53163	.63275	.77983	1.18250
34	.46426	.52163	.60482	.76331	1.18852
35	.45729	.52079	.60769	.78193	1.21168
36	.45770	.51452	.60661	.76218	1.13499
37	.46177	.52033	.61215	.76645	1.12664
38	.46404	.52357	.61461	.76417	1.13875
39	.45700	.51812	.60254	.76641	1.17802
40	.46694	.53085	.61657	.77795	1.15300
41	.45551	.51705	.61321	.76581	1.15760
42	.47180	.53721	.62564	.79213	1.24298
43	.46384	.51566	.60146	.75147	1.08809
44	.47371	.54434	.64197	.80843	1.16483
45	.45976	.52153	.61198	.77735	1.16194
46	.46785	.52882	.62071	.76338	1.21628
47	.46490	.52129	.60793	.77487	1.12421
48	.47582	.53170	.61815	.77982	1.17364
49	.48063	.54400	.63094	.78997	1.19872
50	.47218	.53726	.63046	.79205	1.16489
51	.47487	.53872	.62897	.78620	1.21199
52	.47148	.53079	.61907	.77005	1.12364
53	.47112	.54007	.63222	.79662	1.15710
54	.46084	.51778	.61277	.75934	1.16752
55	.47508	.53724	.63276	.77438	1.18181
56	.46565	.52901	.62268	.78599	1.17529
57	.45185	.51522	.60231	.78654	1.15393
58	.46904	.53178	.60645	.77380	1.13277
59	.47571	.54371	.62089	.76709	1.21326
60	.47305	.53179	.61611	.78893	1.19363

TABLE IV

Powers for Testing H_0: Population Is Normal, When Population Is Exponential

	Actual Population:	Exponential				

n	Calculation method * =straightforward ** =reflection	.20	.15	.10	.05	.01
	Statistic: Kolmogorov-Smirnov					
10	*	.5710	.5120	.4318	.3208	.1612
10	**	.3670	.2914	.2206	.1388	.0390
25	*	.8914	.8528	.7960	.6882	.4536
25	**	.5992	.5262	.4474	.3216	.1532
40	*	.9828	.9752	.9556	.9074	.7204
40	**	.7716	.7120	.6318	.5140	.3202
60	*	.9994	.9984	.9960	.9838	.9312
60	**	.9100	.8752	.8196	.7226	.5176
	Statistic: Anderson-Darling					
10	*	.6688	.6120	.5282	.4120	.2356
10	**	.3782	.3104	.2396	.1668	.0616
25	*	.9668	.9550	.9328	.8854	.7244
25	**	.6656	.6036	.5152	.3932	.1928
40	*	.9980	.9962	.9928	.9840	.9444
40	**	.8428	.7924	.7206	.6016	.3730
60	*	1.0000	1.0000	.9998	.9994	.9948
60	**	.9412	.9172	.8798	.7898	.5556
	Statistic: Cramer-von Mises					
10	*	.6306	.5764	.4944	.3842	.1970
10	**	.3502	.2818	.2134	.1318	.0494
25	*	.9400	.9214	.8910	.8238	.6552
25	**	.5932	.5184	.4194	.2884	.1328
40	*	.9950	.9906	.9838	.9656	.8952
40	**	.7432	.6738	.5810	.4528	.2354
60	*	.9998	.9998	.9992	.9980	.9876
60	**	.8790	.8348	.7588	.6228	.3582

TABLE V

Powers for Testing H_0: Population is Normal, When Population Is Cauchy

	Actual Population: Cauchy					
	Statistic: Kolmogorov-Smirnov					
n	Calculation method * = straightforward ** = reflection	Powers at α-levels				
		.20	.15	.10	.05	.01
10	*	.7306	.6998	.6532	.5884	.4660
10	**	.6442	.6010	.5476	.4732	.3312
25	*	.9558	.9452	.9298	.9000	.8385
25	**	.9532	.9432	.9328	.9066	.8372
40	*	.9918	.9888	.9862	.9766	.9498
40	**	.9934	.9922	.9898	.9838	.9682
60	*	.9994	.9990	.9986	.9970	.9926
60	**	1.0000	.9998	.9994	.9990	.9976
	Statistic: Anderson-Darling					
10	*	.7452	.7132	.6688	.6082	.5010
10	**	.6478	.6042	.5634	.5064	.3838
25	*	.9662	.9610	.9524	.9358	.8870
25	**	.9618	.9538	.9414	.9246	.8708
40	*	.9946	.9936	.9924	.9884	.9740
40	**	.9950	.9942	.9930	.9892	.9788
60	*	1.0000	1.0000	.9998	.9994	.9976
60	**	1.0000	1.0000	1.0000	.9998	.9990
	Statistic: Cramer-von Mises					
10	*	.7436	.7090	.6658	.6104	.4816
10	**	.6456	.6088	.5582	.4878	.3778
25	*	.9644	.9578	.9468	.9294	.8826
25	**	.9608	.9508	.9404	.9196	.8724
40	*	.9950	.9936	.9922	.9874	.9730
40	**	.9954	.9936	.9926	.9896	.9782
60	*	1.0000	.9998	.9998	.9992	.9980
60	**	1.0000	1.0000	1.0000	.9996	.9986

TABLE VI

Powers for Testing H_0: Population Is Normal,
When Population Is χ^2 (1 d.f.)

		Actual Population: χ^2 (1 d.f.)				
		Statistic: Kolmogorov-Smirnov				
n	Calculation method *=straightforward **=reflection	Powers at α-levels				
		.20	.15	.10	.05	.01
10	*	.7850	.7366	.6608	.5420	.3430
10	**	.5762	.4994	.4046	.2854	.1262
25	*	.9904	.9860	.9738	.9492	.8484
25	**	.9262	.9000	.8616	.7692	.5636
40	*	1.0000	1.0000	.9996	.9990	.9864
40	**	.9896	.9844	.9726	.9490	.8570
60	*	1.0000	1.0000	1.0000	1.0000	.9998
60	**	.9996	.9990	.9984	.9950	.9775
		Statistic: Anderson-Darling				
10	*	.8818	.8418	.7832	.6822	.4952
10	**	.5796	.5000	.4124	.2988	.1378
25	*	.9994	.9992	.9980	.9930	.9662
25	**	.9314	.9066	.8600	.7654	.5228
40	*	1.0000	1.0000	1.0000	1.0000	.9994
40	**	.9918	.9862	.9746	.9430	.8238
60	*	1.0000	1.0000	1.0000	1.0000	1.0000
60	**	.9996	.9994	.9986	.9956	.9650
		Statistic: Cramer-von Mises				
10	*	.8512	.8122	.7444	.6476	.4356
10	**	.5436	.4754	.3780	.2508	.1126
25	*	.9978	.9964	.9922	.9818	.9398
25	**	.9042	.8670	.7960	.6794	.4422
40	*	1.0000	1.0000	1.0000	.9998	.9982
40	**	.9798	.9690	.9434	.8926	.7218
60	*	1.0000	1.0000	1.0000	1.0000	1.0000
60	**	.9990	.9984	.9952	.9834	.9044

TABLE VII

Powers for Testing H_0: Population Is Normal,
When Population Is χ^2 (4 d.f.)

Actual Population: χ^2(4 d.f.)						
Statistic: Kolmogorov-Smirnov						
n	Calculation method *=straightforward **=reflection	Powers at α-levels				
		.20	.15	.10	.05	.01
10	*	.4138	.3488	.2716	.1806	.0708
10	**	.2694	.2064	.1406	.0740	.0112
25	*	.6566	.5974	.5146	.3872	.1932
25	**	.3470	.2746	.2126	.1190	.0394
40	*	.8340	.7910	.7248	.6036	.3548
40	**	.4106	.3362	.2582	.1612	.0652
60	*	.9348	.9132	.8640	.7648	.5392
60	**	.4840	.4092	.3228	.2176	.0922
Statistic: Anderson-Darling						
10	*	.4768	.4020	.3068	.2162	.0980
10	**	.2794	.2168	.1586	.0920	.0208
25	*	.7948	.7520	.6884	.5804	.3412
25	**	.4296	.3616	.2764	.1792	.0634
40	*	.9394	.9164	.8820	.8102	.6198
40	**	.5370	.4698	.3910	.2730	.1308
60	*	.9888	.9824	.9682	.9320	.8075
60	**	.6458	.5812	.5006	.3720	.1894
Statistic: Cramer-von Mises						
10	*	.4398	.3696	.2870	.1976	.0784
10	**	.2630	.2060	.1438	.0774	.0160
25	*	.7402	.6830	.6122	.4928	.2852
25	**	.3698	.3014	.2194	.1308	.0494
40	*	.9086	.8758	.8244	.7288	.5248
40	**	.4200	.3536	.2758	.1830	.0770
60	*	.9752	.9622	.9416	.8946	.7506
60	**	.5152	.4384	.3506	.2330	.0922

TABLE VIII

Powers for Testing H_O: Population Is Normal,
When Population Is Double Exponential

Actual Population: Double Exponential						
Statistic: Kolmogorov-Smirnov						
n	Calculation method *=straightforward **=reflection	Powers at α-levels				
		.20	.15	.10	.05	.01
10	*	.3604	.3030	.2376	.1572	.0646
10	**	.2874	.2306	.1698	.1090	.0330
25	*	.5084	.4402	.3618	.2566	.1196
25	**	.5098	.4528	.3912	.2862	.1354
40	*	.6376	.5858	.5114	.3852	.1820
40	**	.6702	.6150	.5440	.4312	.2616
60	*	.7536	.7036	.6264	.4816	.2664
60	**	.8088	.7646	.7046	.6004	.4020
Statistic: Anderson-Darling						
10	*	.3728	.3170	.2414	.1636	.0664
10	**	.2716	.2172	.1708	.1140	.0368
25	*	.5566	.5136	.4418	.3440	.1742
25	**	.5358	.4792	.4094	.3258	.1724
40	*	.6958	.6444	.5794	.4846	.2914
40	**	.7012	.6558	.5970	.4994	.3180
60	*	.8072	.7627	.6978	.5932	.3784
60	**	.8316	.8016	.7558	.6724	.4776
Statistic: Cramer-von Mises						
10	*	.3696	.3130	.2416	.1592	.0546
10	**	.2770	.2252	.1708	.1104	.0428
25	*	.5456	.4844	.4206	.3106	.1608
25	**	.5364	.4838	.4098	.3208	.1802
40	*	.6896	.6408	.5688	.4544	.2646
40	**	.6878	.6476	.5890	.4986	.3210
60	*	.8014	.7626	.7060	.6124	.3964
60	**	.8390	.8018	.7494	.6678	.4726

TABLE IX

Powers for Testing H_0: Population Is Normal, When Population is Logistic

Actual Population: Logistic						

Statistic: Kolmogorov-Smirnov

n	Calculation method *=straightforward **=reflection	.20	.15	.10	.05	.01
10	*	.2486	.1990	.1418	.0874	.0252
10	**	.2328	.1752	.1200	.0676	.0118
25	*	.2670	.2150	.1494	.0876	.0244
25	**	.2774	.2174	.1620	.0932	.0292
40	*	.2958	.2450	.1798	.1044	.0312
40	**	.3052	.2452	.1826	.1094	.0412
60	*	.3306	.2736	.1990	.1130	.0342
60	**	.3542	.2990	.2306	.1588	.0582

Statistic: Anderson-Darling

n		.20	.15	.10	.05	.01
10	*	.2498	.1950	.1374	.0806	.0244
10	**	.2158	.1634	.1126	.0638	.0128
25	*	.2970	.2562	.1982	.1274	.0434
25	**	.2866	.2318	.1660	.1058	.0370
40	*	.3378	.2750	.2208	.1536	.0556
40	**	.3264	.2734	.2156	.1412	.0600
60	*	.3848	.3248	.2458	.1652	.0596
60	**	.3996	.3460	.2784	.1916	.0860

Statistic: Cramer-von Mises

n		.20	.15	.10	.05	.01
10	*	.2358	.1892	.1326	.0780	.0188
10	**	.2162	.1692	.1164	.0602	.0172
25	*	.2792	.2208	.1712	.1020	.0346
25	**	.2858	.2284	.1622	.0980	.0388
40	*	.3200	.2612	.2002	.1290	.0418
40	**	.3104	.2590	.1938	.1296	.0524
60	*	.3552	.3048	.2384	.1578	.0592
60	**	.3816	.3276	.2582	.1776	.0764

TABLE X

Powers for Testing H_O: Population is Normal,
When Population is Student's t (3 d.f.)

Actual Population: Student's t (3 d.f.)						

Statistic: Kolmogorov-Smirnov

n	Calculation method *=straightforward **=reflection	Powers at α-level				
		.20	.15	.10	.05	.01
10	*	.3610	.3066	.2500	.1726	.0838
10	**	.3028	.2498	.1836	.1174	.0378
25	*	.5138	.4596	.3866	.3004	.1700
25	**	.5232	.4646	.4066	.3200	.1750
40	*	.6324	.5892	.5132	.4140	.2482
40	**	.6646	.6124	.5468	.4632	.3288
60	*	.7512	.7024	.6356	.5282	.3632
60	**	.7992	.7628	.7162	.6354	.4752

Statistic: Anderson-Darling

n	method	.20	.15	.10	.05	.01
10	*	.3778	.3214	.2612	.1884	.0990
10	**	.3026	.2524	.1994	.1356	.0552
25	*	.5734	.5366	.4798	.4006	.2530
25	**	.5636	.5124	.4482	.3712	.2374
40	*	.7054	.6634	.6148	.5352	.3788
40	**	.7124	.6734	.6226	.5446	.4076
60	*	.8202	.7868	.7376	.6614	.5040
60	**	.8398	.8146	.7820	.7130	.5766

Statistic: Cramer-von Mises

n	method	.20	.15	.10	.05	.01
10	*	.3622	.3122	.2524	.1810	.0834
10	**	.3020	.2490	.1942	.1242	.0568
25	*	.5470	.4964	.4376	.3506	.2266
25	**	.5500	.4998	.4310	.3518	.2302
40	*	.6866	.6418	.5832	.4842	.3332
40	**	.6924	.6476	.5904	.5192	.3848
60	*	.7962	.7618	.7142	.6454	.4892
60	**	.8256	.7964	.7540	.6874	.5464

TABLE XI

Powers for Testing H_0: Population Is Normal,
When Population is Uniform (Continuous)

Actual Population: Uniform (Continuous)						
Statistic: Kolmogorov-Smirnov						
n	Calculation method *=straightforward **=reflection	Powers at α-level				
		.20	.15	.10	.05	.01
10	*	.2688	.2112	.1420	.0724	.0128
10	**	.3418	.2696	.1946	.1116	.0250
25	*	.3704	.2998	.2156	.1172	.0294
25	**	.5718	.4900	.4078	.2596	.0856
40	*	.5284	.4482	.3424	.1978	.0454
40	**	.7204	.6546	.5542	.4080	.1886
60	*	.6800	.6012	.4918	.3038	.0952
60	**	.8790	.8328	.7586	.6162	.3340
Statistic: Anderson-Darling						
10	*	.3160	.2428	.1596	.0768	.0128
10	**	.3584	.2828	.2226	.1386	.0250
25	*	.5570	.4890	.3866	.2500	.0690
25	**	.6814	.6122	.5190	.3746	.1384
40	*	.7572	.6874	.5910	.4414	.1780
40	**	.8636	.8182	.7506	.6158	.3442
60	*	.9178	.8816	.8038	.6670	.3310
60	**	.9708	.9540	.9218	.8478	.6038
Statistic: Cramer-von Mises						
10	*	.2880	.2194	.1450	.0690	.0102
10	**	.3338	.2798	.2098	.1216	.0322
25	*	.4724	.3856	.2890	.1736	.0476
25	**	.6244	.5530	.4528	.3182	.1250
40	*	.6602	.5806	.4754	.3120	.1040
40	**	.7842	.7226	.6434	.5120	.2602
60	*	.8194	.7732	.6950	.5416	.2366
60	**	.9262	.8960	.8454	.7334	.4588

TABLE XII

Critical Values Used in the Power Study for
the Unmodified Kolmogorov-Smirnov Statistic

n	α-level				
	.20	.15	.10	.05	.01
10	.21595	.22547	.23857	.25841	.29564
25	.14388	.15070	.15990	.17370	.19991
40	.11442	.11937	.12631	.13792	.16200
60	.09443	.09871	.10489	.11506	.13275

TABLE XIII

Critical Values Used in the Power Study for
the Unmodified Anderson-Darling Statistic

n	α-level				
	.20	.15	.10	.05	.01
10	.46452	.51170	.58377	.68950	.90866
25	.49224	.53019	.59532	.70333	.98629
40	.50112	.55038	.61634	.72494	.99653
60	.50662	.55866	.63651	.76620	1.06946

TABLE XIV

Critical Values Used in the Power Study for
the Unmodified Cramer-von Mises Statistic

n	α-level				
	.20	.15	.10	.05	.01
10	.07821	.08720	.10042	.12058	.17031
25	.08110	.09046	.10300	.12522	.17525
40	.08059	.08940	.10275	.12604	.17781
60	.08221	.09052	.10270	.12414	.17726

TABLE XV

Critical Values Used in the Power Study for
the Modified Cramer-von Mises Statistic

n	α-level .20	.15	.10	.05	.01
10	.07142	.08137	.09636	.12401	.18158
25	.07181	.08269	.09921	.12744	.19170
40	.07477	.08630	.10250	.12890	.19513
60	.07370	.08433	.10076	.13065	.20545

happen.

The results using these three additional distribu-
tions are included in Tables IX, X, and XI, respectively.

Critical Values Used. The critical values used in
the power study for the modified K-S and A-D statistics are
the ones in Tables II and III at n = 10, 25, 40, and 60.
The critical values for the CVM statistic modified by reflec-
tion are in Table XV.

The critical values used for the straightforward
calculation of the statistics are in Tables XII, XIII, and
XIV. Tables XII through XV were all generated using 5000
samples. This last set of tables is included for informa-
tional purposes only. The author does not claim that inter-
polation can be done for sample sizes not shown.

Summary

This chapter is essentially a collection of tables

58

with explanatory comments. The tables display the important results of this research effort. The next chapter is a short discussion of the conclusions to be drawn from these results and of any implications for further research.

V. Conclusions and Recommendations

This chapter is a presentation of the author's con-
clusions concerning his research and his recommendations for
further research with the modified statistics. First, a
review of Schuster's (1973; 1975) ideas which apply here
will be presented along with a restatement of the general
research hypothesis. Second, conclusions about how the
actual results compare with the hypothesized results are
presented. In the same section, conclusions are stated con-
cerning the "best" plotting position and the "best" number
of samples to use for the bootstrap technique of determining
critical values.

Review

The purpose of this research has been to test the
technique of reflecting data points about the arithmetic
mean before calculating previously developed goodness of fit
test statistics. This concept was motivated by work done by
Schuster (1973; 1975). The idea that samples can be reflected
about the mean is his. He used the concept to develop a dif-
ferent statistic than the ones which are presented and studied
in this paper. Schuster, however, predicted that the reflec-
tion concept would be helpful when testing within the set of
symmetrical distributions (Schuster, 1973). He also showed
that when the parameters are unknown and when testing within

the set of symmetrical distributions, the statistic he developed would be asymptotically better than statistics calculated without incorporating some kind of reflection technique (Schuster, 1975). Schuster further demonstrated that his statistic would not show improvement when testing a symmetrical versus a non-symmetrical distribution (Schuster, 1973).

Since the statistics studied here are also based upon the same type of reflection, it was expected that using the normal as the hypothesized distribution, 1) improved power would be evident when deviates from other symmetrical distributions were tested, 2) when improved power was evident, it would be more evident as sample size increased (i.e., asymptotically better), and 3) no improvement would be evident in powers generated against the non-symmetric distributions.

The general hypothesis used to guide the research was stated in Chapter I:

> For the normal distribution, the K-S, A-D, and CVM statistics, modified by calculation after doubling the sample by reflecting data points about the sample mean, provide more powerful tests of goodness of fit than do the same statistics calculated without reflection.

Conclusions

Primary Research. Although the three new statistics

61

tested in this thesis are not identical to Schuster's, the predictions made based upon his work are valid. The powers calculated for symmetrical alternatives to the normal are asymptotically greater for the three modified statistics than for the corresponding unmodified statistics. Also, the powers for the three new statistics, when calculated for non-symmetrical alternatives to the normal, are lower than for their unmodified counterparts. This can be seen in the power study tables of Chapter IV.

The general research hypothesis is only partially valid. The modified statistics are not universally of higher power than their unmodified counterparts. Higher powers are evident only for larger sample sizes ($n \geq 25$ in some instances, $n \geq 40$ in most instances) when continuous symmetrical alternatives are tested. The only alternative distribution for which the modified statistics display higher power for all sample sizes is the continuous uniform. Thus, the research hypothesis is false with (continuous) non-symmetrical alternative distributions, partially true for (continuous) symmetrical alternatives, and true when the alternative distribution tested is the (continuous) uniform.

The problem implied by these conclusions is that the applicability of the statistical tables generated is limited. It is the author's conclusion that the tables are useful when it has already been determined (or is highly suspected) that the population from which the sample is drawn is distributed symmetrically. Even with symmetrical

distributions, the tables are only useful for larger sample sizes. The only distributions for which the power with the modified statistics is substantially greater are the double exponential, Student's t with three degrees of freedom, and the uniform.

Another thing the analyst should consider before using these new statistics is whether the significant losses of power against non-symmetrical distributions are worth trading for the much smaller increases in power against the symmetrical distributions. It must be remembered that H_A (the alternative hypothesis) is that the sample is not from a normal population. If he has no knowledge of the population from which the sample is drawn, the analyst could sacrifice substantial power by using these modified statistics.

Finally, the power study tables have been integrated into Table XVI. The statistic which had the highest power, for a given sample size and α-level, have been listed opposite the alternative distribution for which the power was calculated. For instance, for the logistic distribution at $\alpha = .20$ and n = 40, the most powerful statistic of the six is the Anderson-Darling, calculated without reflecting the sample. Throughout Table XVI, an "S" in parentheses indicates straightforward (unmodified) calculation of the statistic. An "R" in parentheses indicates calculation of the statistic after reflection. The non-symmetrical distributions tested are not included in the table because, for all sample sizes and all α-levels, the unmodified Anderson-Darling statistic

63

TABLE XVI

The Statistics with Highest Power When Critical
Values for the Normal Are Tested Using Various
Symmetrical Alternative Distributions

Distribution Tested	n	α-level				
		.20	.15	.10	.05	.01
Uniform	10	A-D(R)	A-D(R)	A-D(R)	A-D(R)	A-D(R)
	25	"	"	"	"	"
	40	"	"	"	"	"
	60	"	"	"	"	"
Logistic	10	A-D(S)	A-D(S)	A-D(S)	A-D(S)	A-D(S)
	25	"	"	"	"	"
	40	"	"	"	"	"
	60	A-D(R)	A-D(R)	A-D(R)	A-D(R)	A-D(R)
Student's t (3 d.f)	10	A-D(S)	A-D(S)	A-D(S)	A-D(S)	A-D(S)
	25	"	"	"	"	"
	40	A-D(R)	A-D(R)	A-D(R)	A-D(R)	A-D(R)
	60	"	"	"	"	"
Cauchy	10	A-D(S)	A-D(S)	A-D(S)	A-D(S)	CVM(S) A-D(S)
	25	"	"	"	"	
	40	CVM(R) A-D(R)	A-D(R)	A-D(R)	A-D(R)	A-D(R)
	60		"	"	"	"
Double Exponential	10	A-D(S)	A-D(S)	CVM(S) A-D(S)	A-D(S)	A-D(S)
	25	"	"		"	CVM(R) CVM(R)
	40	A-D(R)	A-D(R)	A-D(R)	A-D(R)	A-D(R)
	60	CVM(R)	"	"	"	

64

is the most powerful for these distributions.

It should be noted that the evident predominance of the Anderson-Darling statistic was the basis for not generating a critical value table for the modified Cramer-von Mises statistic.

Ancillary Research Issues. The author's conclusions about the other issues tested are made apparent in the decisions discussed in Chapter IV. As far as determination of the "best" plotting position to use with the bootstrap technique is concerned, the conclusion is that when large numbers of statistics are to be plotted, it makes no difference which of the three plotting positions is used.

The conclusion that 5000 (versus 150, 300, 500, and 1000) samples was the number of samples to use to generate critical values is sufficiently explained in Chapter IV.

Recommendations for Further Research

The power study done for this thesis is extensive and the conclusions, thus, are based on rather thorough research. The author sees no apparent reason to make further studies of this new technique with the normal distribution.

However, the results of the power study when the continuous uniform distribution is used are interesting. The power increase that results is quite substantial. The powers demonstrated are better than for any of the statistics tested by Green and Hegazy (1976). Perhaps, if the technique of reflection is applied to the same statistics to generate

65

critical values for the continuous uniform, the resultant powers for the uniform might be very high. This might, at least, be the case when samples from symmetrical distributions are tested.

The only other suggestion concerns the number of samples to use with the bootstrap technique. The decision to use 5000 samples rather than investigate alternative numbers between 1000 and 5000 samples was one of expedience. Before the bootstrap technique is again used to find critical values, numbers of samples greater than 1000 and less than 5000 should be examine for consistency at the $\alpha = .01$ level of significance. Some savings of computer resource may still be possible.

Bibliography

Anderson, T.W., and D.A. Darling. "Asymptotic Theory of Certain 'Goodness of Fit' Criteria Based on Stochastic Processes," Annals of Mathematical Statistics, 24:193-212 (1952).

———. "A Test of Goodness of Fit," Journal of the American Statistical Association, 49: 765-769 (December 1954).

Easterling, R.G. "Goodness of Fit and Parameter Estimation," Technometrics, 18: 1-9 (February 1976).

Efron, B. "Bootstrap Methods: Another Look at the Jackknife," The Annals of Statistics, 7: 1-26 (January 1979).

Green, J.R., and Y.A.S. Hegazy. "Powerful Modified-EDF Goodness-of-Fit Tests," Journal of the American Statistical Association, 71: 204-209 (March 1976).

Johnston, J.W., Jr. "A Modified Double Monte Carlo Technique to Approximate Reliability Confidence Limits of Systems With Components Characterized by the Weibull Distribution." Unpublished MS thesis. Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1980.

Kapur, K.C., and L.R. Lamberson. Reliability in Engineering Design. New York: John Wiley & Sons, 1977.

Kimball, B.F. "On the Choice of Plotting Positions on Probability Paper," Journal of the American Statistical Association, 69: 730-737 (September 1974).

Lilliefors, H.W. "On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown," Journal of the American Statistical Association, 62: 399-402 (June 1967).

Littel, R.C., J.T. McClave, and W.W. Offen. "Goodness-of-Fit Tests for the Two Parameter Weibull Distribution," Communications in Statistics--Simulation and Computation, B8: 257-269 (1979).

Massey, F.J., Jr. "The Kolmogorov-Smirnov Test for Goodness of Fit," Journal of the American Statistical Association, 46: 68-78 (March 1951).

Mendenhall, W., and R.L. Schaeffer. Mathematical Statistics With Applications. North Scituate, Ma.: Duxbury Press, 1973.

Schuster, E.F. "Estimating the Distribution Function of a Symmetric Distribution," Biometrika, 62: 631-635 (1975).

_____. "On the Goodness of Fit Problem for Continuous Symmetric Distributions," Journal of the American Statistical Association, 68: 713-715 (September 1973).

(For corrections to Schuster (1973), see "Corrigenda," Journal of the American Statistical Association, 69: 288 (March 1974).

Stephens, M.A. "EDF Statistics for Goodness of Fit and Some Comparisons," Journal of the American Statistical Association, 69: 730-737 (September 1974).

APPENDIX A

COMPAR

The computer code for comparing the median rank, the modified step rank, and the average of the mean and mode ranks as plotting positions is included in the following three pages.

THE PURPOSE OF THIS PROGRAM IS TO DEMONSTRATE THE DIFFERENCES AMONG THREE DIFFERENT PLOTTING POSITIONS. THE POSITIONS BEING COMPARED ARE

$$\text{MEDIAN RANKS} = \frac{I - 0.3}{N + 0.4}$$

$$\text{MODIFIED STEP RANKS} = \frac{I - 0.5}{N}, \text{ AND}$$

$$\text{AN AVERAGE OF MEAN AND MODE RANKS} = \frac{\frac{I}{N+1} + \frac{I-1}{N-1}}{2.0}$$

THIS PROGRAM PLACES VALUES FOR EACH PLOTTING POSITION FOR I=1 TO N INTO ARRAYS. THESE VALUES ARE THEN COMPARED ONE BY ONE.

THE OUTPUT IS A LISTING OF N VALUES FOR EACH TYPE OF PLOTTING POSITION. ALSO LISTED ARE THE ABSOLUTE DIFFERENCES AMONG THEM.

A LISTING OF VARIABLE DEFINITIONS FOLLOWS:

I = 1,2,3,...,N
N = TOTAL NUMBER OF POINTS BEING PLOTTED
P,Q,S, AND K ARE USED AS PROGRAM INDECES
A(I) = AN ARRAY OF MEDIAN RANKS
B(I) = AN ARRAY OF MODIFIED STEP RANKS
C(I) = AN ARRAY OF AVERAGES OF MODE AND MEAN RANKS
D(I) = A(I) - B(I)

70

```
      F(I) = B(I) - C(I)
      F(I) = A(I) - C(I)
      R IS USED FOR AN INTERIM COMPUTATION

      PROGRAM COMPAR
      INTEGER  I,N,P,Q,S,K,J
      REAL     A(300),B(300),C(300),D(300),E(300),F(300),
     +         IREAL,NREAL,R
      N=150
      DO 100 I = 1,N
      IREAL = REAL (I)
      NREAL = REAL (N)
      A(I) = (IREAL - 0.3)/(NREAL + 0.4)
      B(I) = (IREAL - 0.5) / NREAL
      C(I) = ((IREAL/(NREAL+1.0)+((IREAL-1.0)/(NREAL-1.0)))/2.0
      D(I) = ABS (A(I) - B(I))
      E(I) = ABS (B(I) - C(I))
      F(I) = ABS (A(I) - C(I))
100   CONTINUE
      P = 1
      Q = 25
      R = NREAL/25.0
      S = INT(R)

      DO 200 J = 1,S
      PRINT '(A)', '1'
      PRINT '(T25,A)', 'LISTING OF PLOTTING POSITIONS'
      PRINT '(T21,A)', 'ALONG WITH ABSOLUTE DIFFERENCES BETWEEN'
      PRINT '(T26,A)', 'THEM TO FIVE DECIMAL PLACES.'
      PRINT '(A,T26,A)','+','',
      PRINT'(//T19,A,T32,A,T45,A,T57,A,T63,A)','MODIFIED',
     +     'AVG MEAN %', 'DELTA', 'DELTA', 'DELTA'
```

71

```fortran
      PRINT "(T3,A,T7,A,T19,A,T32,A,T45,A,T56,A,T68,A)",
     +   "I ","MEDIAN(M)","STEP(YS)","MODE(AMM)",
     +   "MS VS M", "AMM VS MS", "M VS AMM"
      PRINT "(A,T3,A,T7,A,T19,A,T32,A,T45,A,T56,A,T68,A)",
     +   "-","--","--------","--------","--------","--------",
     +   "-------","--------","--------"
      DO 300 K = P,Q
        PRINT "(T2,I3,T8,F6.5,T21,F6.5,T34,F6.5,
     +     T46,F6.5,T57,F6.5,T70,F6.5)",
     +     K,A(K),B(K),C(K),D(K),E(K),F(K)
300   CONTINUE
      IF (J.NE.S) THEN
        P = P + 25
        Q = Q + 25
      ELSEIF (MOD(N,25).GT.0) THEN
        P = P + 25
        Q = Q + MOD(N,25)
      ELSE
        GOTO 400
      ENDIF
200   CONTINUE
400   CONTINUE
      END
```

72

APPENDIX B

Results of COMPAR

The results of program COMPAR are included in the
following six pages. These particular results are for when
there are 150 points to be plotted.

LISTING OF PLOTTING POSITIONS
ALONG WITH ABSOLUTE DIFFERENCES BETWEEN
THEM TO FIVE DECIMAL PLACES.

I	MEDIAN[M]	MODIFIED SIERIN[S1]	AVG MEAN & MODE[AIM]	DELTA MS VS M	DELTA AIM VS MS	DELTA M VS AIM
1	.00165	.00333	.00331	.00132	.00002	.00134
2	.01130	.01000	.00998	.00130	.00002	.00132
3	.01795	.01667	.01665	.00129	.00002	.00131
4	.02460	.02333	.02331	.00127	.00002	.00129
5	.03125	.03000	.02998	.00125	.00002	.00127
6	.03790	.03667	.03665	.00123	.00002	.00125
7	.04455	.04333	.04331	.00121	.00002	.00123
8	.05120	.05000	.04998	.00120	.00002	.00122
9	.05785	.05667	.05665	.00118	.00002	.00120
10	.06445	.06333	.06331	.00116	.00002	.00118
11	.07114	.07000	.06998	.00114	.00002	.00116
12	.07779	.07667	.07665	.00113	.00002	.00114
13	.08444	.08333	.08331	.00111	.00002	.00113
14	.09109	.09000	.08998	.00109	.00002	.00111
15	.09774	.09667	.09665	.00107	.00002	.00109
16	.10439	.10333	.10332	.00105	.00002	.00107
17	.11104	.11000	.10998	.00104	.00002	.00105
18	.11769	.11667	.11665	.00102	.00002	.00104
19	.12434	.12333	.12332	.00100	.00002	.00102
20	.13098	.13000	.12998	.00098	.00002	.00100
21	.13763	.13667	.13665	.00097	.00002	.00098
22	.14428	.14333	.14332	.00095	.00002	.00096
23	.15093	.15000	.14998	.00093	.00002	.00095
24	.15758	.15667	.15665	.00091	.00002	.00093
25	.16423	.16333	.16332	.00090	.00001	.00091

LISTING OF PLOTTING POSITIONS
ALONG WITH ABSOLUTE DIFFERENCES BETWEEN
THEM TO FIVE DECIMAL PLACES.

I	MEDIAN(M)	MODIFIED SIEG(MS)	AVG MEAN & MODE(AMM)	DELTA MS-M	DELTA AMM-MS	DELTA M-VS-AMM
26	.17966	.17000	.16999	.00188	.00161	.00189
27	.17753	.17667	.17665	.00086	.00001	.00087
28	.18418	.18333	.18332	.00084	.00001	.00086
29	.19082	.19000	.18999	.00082	.00001	.00084
30	.19747	.19567	.19665	.00081	.00001	.00082
31	.20412	.20333	.20332	.00079	.00001	.00080
32	.21077	.21000	.20999	.00077	.00001	.00078
33	.21742	.21667	.21665	.00075	.00001	.00077
34	.22437	.22333	.22332	.00074	.00001	.00075
35	.27072	.23000	.22999	.00072	.00001	.00073
36	.23737	.23667	.23665	.00070	.00001	.00071
37	.24402	.24333	.24332	.00068	.00001	.00069
38	.25066	.25000	.24999	.00066	.00001	.00068
39	.25731	.25657	.25666	.00065	.00001	.00066
40	.26396	.26333	.26332	.00063	.00001	.00064
41	.27061	.27000	.26999	.00061	.00001	.00062
42	.27726	.27657	.27666	.00059	.00001	.00060
43	.28391	.28333	.28332	.00058	.00001	.00059
44	.29056	.29000	.28999	.00056	.00001	.00057
45	.29721	.29657	.29666	.00054	.00001	.00055
46	.30386	.30333	.30332	.00052	.00001	.00053
47	.31051	.31000	.30999	.00051	.00001	.00051
48	.31715	.31667	.31666	.00049	.00001	.00050
49	.32380	.32333	.32333	.00047	.00001	.00048
50	.33045	.33000	.32999	.00045	.00001	.00046

LISTING OF PLOTTING POSITIONS
ALONG WITH ABSOLUTE DIFFERENCES BETWEEN
THEM TO FIVE DECIMAL PLACES.

I	MEDIAN(M)	MODIFIED SIEB(MS)	AVG MEAN & MODE(AMM)	DELTA MS_VS_M	DELTA AMM_VS_MS	DELTA M_VS_AMM
51	.33710	.33667	.33666	.00043	.00001	.00044
52	.34375	.34333	.34333	.00042	.00001	.00042
53	.35040	.35000	.34999	.00040	.00001	.00041
54	.35705	.35667	.35666	.00038	.00001	.00039
55	.36370	.36333	.36333	.00036	.00001	.00037
56	.37035	.37000	.36999	.00035	.00001	.00035
57	.37699	.37667	.37666	.00033	.00001	.00033
58	.38364	.38333	.36333	.00031	.00001	.00032
59	.39029	.39000	.39000	.00029	.00003	.00030
60	.39694	.39667	.39666	.00027	.00000	.00028
61	.40359	.40333	.40333	.00026	.00000	.00026
62	.41024	.41000	.41000	.00024	.00003	.00024
63	.41689	.41667	.41666	.00022	.00000	.00023
64	.42354	.42333	.42333	.00020	.00000	.00021
65	.43019	.43000	.43000	.00019	.00000	.00019
66	.43684	.43667	.43666	.00017	.00000	.00017
67	.44348	.44333	.44333	.00015	.00000	.00015
68	.45013	.45000	.45000	.00013	.00000	.00014
69	.45678	.45667	.45666	.00012	.00000	.00012
70	.46343	.46333	.46333	.00011	.00000	.00010
71	.47008	.47000	.47000	.00008	.00000	.00008
72	.47673	.47667	.47667	.00006	.00000	.00006
73	.48338	.48333	.48333	.00004	.00000	.00005
74	.49003	.49000	.49000	.00003	.00000	.00003
75	.49668	.49667	.49667	.00001	.00000	.00001

LISTING OF PLOTTING POSITIONS
ALONG WITH ABSOLUTE DIFFERENCES BETWEEN
THEM TO FIVE DECIMAL PLACES.

I.	MEDIAN(I)	MODIFIED STEP(I/S)	AVG MEAN MODE(4(I/H))	DELTA MS_VS_M	DELTA MM_VS_MS	DELTA M_VS_MM
76	.50332	.50333	.50333	.00001	.00000	.00001
77	.51597	.51000	.51000	.00003	.00000	.00003
78	.51562	.51667	.51667	.00004	.00000	.00005
79	.52327	.52333	.52333	.00006	.00000	.00006
80	.52992	.53000	.53000	.00008	.00000	.00008
81	.53657	.53667	.53667	.00010	.00000	.00010
82	.54322	.54333	.54334	.00012	.00000	.00012
83	.54987	.55000	.55000	.00013	.00000	.00014
84	.55652	.55667	.55667	.00015	.00000	.00015
85	.56316	.56333	.56334	.00017	.00000	.00017
86	.56981	.57000	.57000	.00019	.00000	.00019
87	.57646	.57667	.57667	.00020	.00000	.00021
88	.58311	.58333	.58334	.00022	.00000	.00023
89	.58976	.59000	.59000	.00024	.00000	.00024
90	.59541	.59557	.59667	.00026	.00000	.00026
91	.60336	.60333	.60334	.00027	.00000	.00028
92	.60971	.61000	.61000	.00029	.00000	.00030
93	.61636	.61667	.61667	.00031	.00001	.00032
94	.62301	.62333	.62334	.00033	.00001	.00033
95	.62965	.63000	.63001	.00035	.00001	.00035
96	.63530	.63667	.63667	.00036	.00001	.00037
97	.64295	.64333	.64334	.00038	.00001	.00039
98	.64360	.65000	.65001	.00040	.00001	.00041
99	.65625	.65667	.65667	.00042	.00001	.00042
100	.66290	.66333	.66334	.00043	.00001	.00044

77

I	MEDIAN(M)	MODIFIED SIEP(MS)	AVG MEAN MODE(AMM)	DELTA MS_VS_M	DELTA AMM_VS_MS	DELTA M_VS_AMM
101	.66955	.67000	.67001	.00045	.00001	.00046
102	.67526	.67667	.67667	.00047	.00001	.00048
103	.68285	.68333	.68334	.00049	.00001	.00050
104	.68949	.69000	.69001	.00051	.00001	.00051
105	.69514	.69667	.69668	.00052	.00001	.00053
106	.70279	.70333	.70334	.00054	.00001	.00055
107	.70944	.71000	.71001	.00056	.00001	.00057
108	.71569	.71667	.71668	.00058	.00001	.00059
109	.72274	.72333	.72334	.00059	.00001	.00060
110	.72939	.73000	.73001	.00061	.00001	.00062
111	.73604	.73667	.73668	.00063	.00001	.00064
112	.74269	.74333	.74334	.00065	.00001	.00066
113	.74934	.75000	.75001	.00066	.00001	.00068
114	.75598	.75667	.75668	.00068	.00001	.00069
115	.76263	.76333	.76335	.00070	.00001	.00071
116	.76928	.77000	.77001	.00072	.00001	.00073
117	.77593	.77667	.77668	.00074	.00001	.00075
118	.78258	.78333	.78335	.00075	.00001	.00077
119	.78923	.79000	.79001	.00077	.00001	.00078
120	.79588	.79667	.79668	.00079	.00001	.00080
121	.80253	.80333	.80335	.00081	.00001	.00082
122	.80918	.81000	.81001	.00082	.00001	.00084
123	.81582	.81667	.81668	.00084	.00001	.00086
124	.82247	.82333	.82335	.00086	.00001	.00087
125	.82912	.83000	.83001	.00088	.00001	.00089

78

LISTING OF PLOTTING POSITIONS ALONG WITH ABSOLUTE DIFFERENCES BETWEEN THEM TO FIVE DECIMAL PLACES.

I.	MEDIAN(M)	MODIFIED STEP(MS)	AVG MEAN & MODE(AMM)	DELTA MS_VS_M	DELTA AMM_VS_MS	DELTA M_VS_AMM
126	.83577	.83667	.83668	.00090	.00001	.00091
127	.84242	.84333	.84335	.00091	.00002	.00093
128	.84937	.85000	.85002	.00093	.00002	.00095
129	.85572	.85667	.85668	.00095	.00002	.00096
130	.86237	.86333	.86335	.00097	.00002	.00098
131	.86902	.87000	.87002	.00098	.00002	.00100
132	.87566	.87667	.87668	.00100	.00002	.00102
133	.88231	.88333	.88335	.00102	.00002	.00104
134	.88896	.89000	.89002	.00104	.00002	.00105
135	.89561	.89667	.89668	.00105	.00002	.00107
136	.90226	.90333	.90335	.00107	.00002	.00109
137	.90891	.91000	.91002	.00109	.00002	.00111
138	.91556	.91667	.91669	.00111	.00002	.00113
139	.92221	.92333	.92335	.00113	.00002	.00114
140	.92866	.93000	.93002	.00114	.00002	.00116
141	.93551	.93667	.93669	.00116	.00002	.00118
142	.94215	.94333	.94335	.00118	.00002	.00120
143	.94880	.95000	.95002	.00120	.00002	.00122
144	.95545	.95667	.95669	.00121	.00002	.00123
145	.96210	.96333	.96335	.00123	.00002	.00125
146	.96875	.97000	.97002	.00125	.00002	.00127
147	.97540	.97667	.97669	.00127	.00002	.00129
148	.98205	.98333	.98335	.00129	.00002	.00131
149	.98870	.99000	.99002	.00130	.00002	.00132
150	.99535	.99667	.99669	.00132	.00002	.00134

79

APPENDIX C

Subroutines

All subroutines written and used by the author are
included in this section. The purpose of each subroutine
is discussed in Chapter III.

```
*************************************************************
SUBROUTINE CVALS(YLOWER,YUPPER,DLOWER,DUPPER,YVALUE,DOUT)
REAL  M,B,YLOWER,DLOWER,YUPPER,LUPPER,YVALUE,DOUT
M = (YUPPER - YLOWER)/(LUPPER - DLOWER)
B = YLOWER - (M * DLOWER)
DOUT = (YVALUE - B) / M
END

*************************************************************
      SUBROUTINE  LILDIF (N,F,DIF)
      REAL   F(*),DIF(*)
      INTEGER   I,N
      DO  100 I = 1,N
          DIF(I) = F(I) -(REAL(I)/REAL(N))
          DIF(I+N) = F(I) - ((REAL(I) - 1.0)/N)
100   CONTINUE
      END

*************************************************************
      SUBROUTINE  DSTAT(M,DIFF,XDIF)
      INTEGER      I,M,M1
      REAL         XDIF,DIFF(*)
      M1 = 2 * M
      XDIF = 0.0
      DO  100  I = 1,M1
          DIFF(I) = ABS(DIFF(I))
          IF (DIFF(I) .GE. XDIF)  XDIF = DIFF(I)
100   CONTINUE
      END

*************************************************************
```

```
*************************************************************

      SUBROUTINE ANDAR(N,U,WSQUAR,COUNT1,COUNT2)
      INTEGER    N,COUNT1,COUNT2,J
      REAL    U(*),SUM,LNUJ,LNUNJ1,WSQUAR,UNJ1,UJ
      COUNT1 = 0
      COUNT2 = 0
      SUM = 0.0
      DO   100 J = 1,N
           UJ = U(J)
           IF (UJ .LE. 0.0) THEN
                UJ = .0001
                COUNT1 = COUNT1 + 1
           ENDIF
           UNJ1 = 1.0 - U(N-J+1)
           IF (UNJ1 .LE. 0.0) THEN
                UNJ1 = .0001
                COUNT2 = COUNT2 + 1
           ENDIF
           LNUJ = LOG(UJ)
           LNUNJ1 = LOG(UNJ1)
           SUM = (((2.0*REAL(J))-1.0) * (LNUJ+LNUNJ1)) + SUM
100   CONTINUE
      WSQUAR = 0.0 - REAL(N) - ((1.0/REAL(N)) * SUM)
      END

*************************************************************

      SUBROUTINE  CVM(N,U,WSQUAR)
      INTEGER      J,N
      REAL          U(*),SUM,WSQUAR,VALUE
      SUM = 0.0
      DO 100 J = 1,N
           VALUE = ((2.0*REAL(J)) - 1.0)/(2.0*REAL(N))
           SUM = SUM + ((U(J) - VALUE) * (U(J) - VALUE))
100   CONTINUE
      WSQUAR = (1.0/(12.0*REAL(N))) + SUM
      END

*************************************************************

      SUBROUTINE  DUBSAM(X,N)
      INTEGER      I,N
      REAL          X(*),XBAR
      XBAR = 0.0
      DO   100  I = 1,N
           XBAR = XBAR + X(I)
100   CONTINUE
      XBAR = XBAR/N
      DO 200  I = 1,N
           X(N+I) = (2.0 * XBAR) - X(I)
200   CONTINUE
      END

*************************************************************
```

81

```
*******************************************************

      SUBROUTINE  ESTPAR(X,N)
      INTEGER   I,N
      REAL      X(*),XSUM,XBAR,S,NMRATR
      XSUM = 0.0
      NMRATR = 0.0
      DO   100  I = 1,N
          XSUM = XSUM + X(I)
100   CONTINUE
      XBAR = XSUM/N
      DO   200  I = 1,N
          NMRATR = NMRATR + ((X(I)-XBAR)*(X(I)-XBAR))
200   CONTINUE
      S = SQRT(NMRATR/(N-1))
      DO   300  I = 1,n
          X(I) = (X(I) - XBAR)/S
300   CONTINUE
      END


*******************************************************

      SUBROUTINE  XPOLAT(N,D)
      INTEGER N,NMIN1,NPLUS1
      REAL    Y1,Y2,D(0:*),DLOWER,DUPPER,XO
      Y1 = 0.5/N
      Y2 = 1.5/N
      DLOWER = D(1)
      DUPPER = D(2)
      CALL CVALS(Y1,Y2,DLOWER,DUPPER,0.0,XO)
      IF (XO .GE. 0.0) THEN
          D(0) = XO
        ELSE
          D(0) = 0.0
      ENDIF
      Y1 = (REAL(N) - 1.5)/N
      Y2 = (REAL(N) - 0.5)/N
      NMIN1 = N - 1
      DLOWER = D(NMIN1)
      DUPPER = D(N)
      CALL CVALS(Y1,Y2,DLOWER,DUPPER,1.0,XO)
      NPLUS1 = N + 1
      D(NPLUS1) = XO
      END


*******************************************************
```

82

```
*************************************************************
      SUBROUTINE CVALUE(D,CVAL80,CVAL85,CVAL90,CVAL95,CVAL99,N)
      INTEGER  I,N,NPLS1
      REAL D(0:*),Y(0:6000),COMP80,COMP90,COMP95,COMP99,
     +  Y79,D79,Y81,D81,DIF90,Y89,Y91,D89,D91,DIF95,DIF80,
     +  Y94,Y96,D94,D96,DIF99,Y98,Y100,D98,D100,DIF85,COMP85,
     +  Y84,D84,Y86,D86,CVAL85,CVAL80,CVAL90,CVAL95,CVAL99
      DO  100  I = 1,N
          Y(I) =(REAL(I) - 0.5)/REAL(N)
100   CONTINUE
      Y(0) = 0.0
      NPLS1 = N + 1
      Y(NPLS1) = 1.0
      COMP80 = 1000.0
      COMP85 = 1000.0
      COMP90 = 1000.0
      COMP95 = 1000.0
      COMP99 = 1000.0
      DO  200  I = NPLS1,0,-1
          IF (Y(I) .LE. 0.75) GOTO 300
          IF (Y(I) .GT. 0.75 .AND. Y(I) .LE. 0.80) THEN
              DIF80 = .80 - Y(I)
              IF (DIF80 .LE. COMP80) THEN
                  COMP80 = DIF80
                  Y79 = Y(I)
                  D79 = D(I)
                  Y81 = Y(I+1)
                  D81 = D(I+1)
              ENDIF
          ELSEIF (Y(I) .GT. .80 .AND. Y(I) .LE. .85) THEN
              DIF85 = .85 - Y(I)
              IF (DIF85 .LE. COMP85) THEN
                  COMP85 = DIF85
                  Y84 = Y(I)
                  D84 = D(I)
                  Y86 = Y(I+1)
                  D86 = D(I+1)
              ENDIF
          ELSEIF (Y(I) .GT. .85 .AND. Y(I) .LE. .90) THEN
              DIF90 = .90 - Y(I)
              IF (DIF90 .LE. COMP90) THEN
                  COMP90 = DIF90
                  Y89 = Y(I)
                  D89 = D(I)
                  Y91 = Y(I+1)
                  D91 = D(I+1)
              ENDIF
```

```fortran
             ELSEIF (Y(I) .GT. .90 .AND. Y(I) .LE. .95)THEN
                 DIF95 = .95 - Y(I)
                 IF (DIF95 .LE. COMP95) THEN
                     COMP95 = DIF95
                     Y94 = Y(I)
                     Y96 = Y(I+1)
                     D94 = D(I)
                     D96 = D(I+1)
                 ENDIF
             ELSEIF (Y(I) .GT. .95 .AND. Y(I) .LE. .99) THEN
                 DIF99 = .99 - Y(I)
                 IF (DIF99 .LE. COMP99) THEN
                     COMP99 = DIF99
                     Y98 = Y(I)
                     Y100 = Y(I+1)
                     D98 = D(I)
                     D100 = D(I+1)
                 ENDIF
             ENDIF
200   CONTINUE
300   CONTINUE
      IF (DIF80 .EQ. 0.0) THEN
          CVAL80 = D79
        ELSE
          CALL  CVALS(Y79,Y81,D79,D81,.80,CVAL80)
      ENDIF
      IF (DIF85 .EQ. 0.0) THEN
          CVAL85 = D84
        ELSE
          CALL  CVALS(Y84,Y86,D84,D86,.85,CVAL85)
      ENDIF
      IF (DIF90 .EQ. 0.0) THEN
          CVAL90 = D89
        ELSE
          CALL  CVALS(Y89,Y91,D89,D91,.90,CVAL90)
      ENDIF
      IF (DIF95 .EQ. 0.0) THEN
          CVAL95 = D94
        ELSE
          CALL CVALS(Y94,Y96,D94,D96,.95,CVAL95)
      ENDIF
      IF (DIF99 .EQ. 0.0) THEN
          CVAL99 = D98
        ELSE
          CALL CVALS(Y98,Y100,D98,D100,.99,CVAL99)
      ENDIF
      END
```

****

84

COMLIL

This program was used to determine the number of samples to use for the bootstrap technique and to validate the logic used to find critical values of the unmodified Kolmogorov-Smirnov statistic.

****

```
      PROGRAM  COMLIL
      INTEGER  SAMSIZ,J,N,K,I,SMPLS1,A,M
      REAL     R(120),DIFFS(240),LILST,Y,P,DSTATS(0:5004),
     +         CV80,CV85,CV95,CV99
      DOUBLE PRECISION  SEED1
      SEED1 = 21478.D0
      DO   400  A = 1,5
          IF (A .EQ. 1) SAMSIZ = 153
          IF (A .EQ. 2) SAMSIZ = 303
          IF (A .EQ. 3) SAMSIZ = 503
          IF (A .EQ. 4) SAMSIZ = 1003
          IF (A .EQ. 5) SAMSIZ = 5003
          DO   500  N = 10,30,10
          PRINT *, 'N = ',N,' AND SAMSIZ = ',SAMSIZ
              DO  100  J = 1,SAMSIZ
                  CALL GONO(SEED1,1,N,N,R)
                  M = N
                  CALL ESTPAR(R,M)
                  CALL VSRTA(R,M)
                  DO  200  K = 1,M
                      Y = R(K)
                      CALL MDNOR(Y,P)
                      R(K) = P
200               CONTINUE
                  CALL LILDIF(M,R,DIFFS)
                  CALL DSTAT(M,DIFFS,LILST)
                  DSTATS(J) = LILST
100           CONTINUE
              DSTATS(0) = 0.0
              SMPLS1 = SAMSIZ + 1
              CALL VSRTA(DSTATS,SMPLS1)
              CALL XPOLAT(SAMSIZ,DSTATS)
          CALL CVALUE (DSTATS,CV80,CV85,CV90,CV95,CV99,SAMSIZ)
```

```
            PRINT *, 'FOR ',SAMSIZ,' D-STATISTICS AT N = ',N
            PRINT *, 'CVAL80 = ', CV80
            PRINT *, 'CVAL85 = ', CV85
            PRINT *, 'CVAL90 = ', CV90
            PRINT *, 'CVAL95 = ', CV95
            PRINT *, 'CVAL99 = ', CV99
            PRINT *
            PRINT *
500      CONTINUE
400   CONTINUE
      END
```

APPENDIX E

TABLE2

This program is typical of the programs used to obtain critical values of the statistics. This particular example is used to find critical values for the Anderson-Darling statistic at sample size, n = 40.

****

```
      PROGRAM   TABLE2
      INTEGER   SAMSIZ,J,N,K,SMPLS1,M,CNT1,CNT2,
     +          COUNT1,COUNT2
      REAL      R(120),Y,P,WSQUAR(0:5004),WSQRD,
     +          CV80,CV85,CV90,CV95,CV99
      DOUBLE PRECISION  SEED1
      SEED1 = 469857936.D0
      COUNT1 = 0
      COUNT2 = 0
      SAMSIZ = 5000
      N = 40
          PRINT *, 'N = ',N,' AND SAMSIZ = ',SAMSIZ
              DO  100  J = 1,SAMSIZ
                  CALL GGNO(SEED1,1,N,N,R)
                  CALL DUBSAM(R,N)
                  M = 2 * N
                  CALL ESTPAR(R,M)
                  CALL VSRTA(R,M)
                  DO  200  K = 1,M
                      Y = R(K)
                      CALL MDNOR(Y,P)
                      R(K) = P
200               CONTINUE
                  CALL ANDAR(M,R,WSQRD,CNT1,CNT2)
                  WSQUAR(J) = WSQRD
                  COUNT1 = COUNT1 + CNT1
                  COUNT2 = COUNT2 + CNT2
100           CONTINUE
```

```
          WSQUAR(0) = 0.0
          SMPLS1 = SAMSIZ + 1
          CALL VSRTA(WSQUAR,SMPLS1)
          CALL XPCLAT(SAMSIZ,WSQUAR)
       CALL CVALUE(WSQUAR,CV80,CV85,CV90,CV95,CV99,SAMSIZ)
    PRINT*,'FOR ',SAMSIZ,' ANDERSON-DARLING STATISTICS AT N=',N
          PRINT *, 'CVAL80 = ', CV80
          PRINT *, 'CVAL85 = ',CV85
          PRINT *, 'CVAL90 = ', CV90
          PRINT *, 'CVAL95 = ', CV95
          PRINT *, 'CVAL99 = ', CV99
          PRINT *
          PRINT *, 'COUNT1 = ',COUNT1
          PRINT *, 'COUNT2 = ',COUNT2
    END
```

APPENDIX F

POWERS

This program is typical of those used in the power study. This particular one is used to find powers for all six statistics when tested against 5000 samples of size, n = 10, from the Cauchy distribution.
****

```
      PROGRAM  POWERS
      INTEGER  NR,J,K,L,M,COUNT(4),POWER(30),CNT1,CNT2,I
      REAL     WK(360),R(120),S(120),T(120),DIFFS(240),
     +         Y,P,LILIES,LILIE2,ANDAR1,ANDAR2,CRVM,
     +         CRVM2,PWR(30)
      DOUBLE PRECISION  SEED1
      SEED1 = 1095785.D0
      DO  600  I = 1,30
          POWER(I) = 0
600   CONTINUE
      DO  800  I = 1,4
          COUNT(I) = 0
800   CONTINUE
      NR = 10
      DO  100  J = 1,5000
          CALL  GGCAY(SEED1,NR,WK,R)
          CALL  VSRTA(R,NR)
          DO  200  K = 1,NR
              S(K) = R(K)
              T(K) = R(K)
200       CONTINUE
          CALL ESTPAR(S,NR)
          CALL DUBSAM(T,NR)
          M = 2 * NR
          CALL ESTPAR(T,M)
          CALL VSRTA(S,NR)
          CALL VSRTA(T,M)
          DO  300 L = 1,M
              Y = T(L)
              CALL MDNOR(Y,P)
              T(L) = P
300       CONTINUE
          DO  400  L = 1,NR
              Y = S(L)
              CALL MDNOR(Y,P)
              S(L) = P
400       CONTINUE
          CALL LILDIF(NR,S,DIFFS)
          CALL DSTAT(NR,DIFFS,LILIES)
          CALL ANDAR(NR,S,ANDAR1,CNT1,CNT2)
```

89

```
COUNT(1) = COUNT(1) + CNT1
COUNT(2) = COUNT(2) + CNT2
CALL CVM(NR,S,CRVM)
CALL LILDIF(N,T,DIFFS)
CALL DSTAT(N,DIFFS,LILIE2)
CALL ANDAR(N,T,ANDAR2,CNT1,CNT2)
COUNT(3) = COUNT(3) + CNT1
COUNT(4) = COUNT(4) + CNT2
CALL CVM(N,T,CRVM2)
IF (LILIES .GT. .21595) THEN
      POWER(1) = POWER(1) + 1
ENDIF
IF (LILIES .GT. .22547) THEN
      POWER(2) = POWER(2) + 1
ENDIF
IF (LILIES .GT. .23857) THEN
      POWER(3) = POWER(3) + 1
ENDIF
IF (LILIES .GT. .25841) THEN
      POWER(4) = POWER(4) + 1
ENDIF
IF (LILIES .GT. .20564) THEN
      POWER(5) = POWER(5) + 1
ENDIF
IF ( LILIE2 .GT. .13452)THEN
      POWER(6) = POWER(6) + 1
ENDIF
IF (LILIE2 .GT. .14278) THEN
      POWER(7) = POWER(7) + 1
ENDIF
IF (LILIE2 .GT. .15309) THEN
      POWER(8) = POWER(8) + 1
ENDIF
IF (LILIE2 .GT. .16858) THEN
      POWER(9) = POWER(9) + 1
ENDIF
IF (LILIE2 .GT. .20295) THEN
      POWER(10) = POWER(10) + 1
ENDIF
IF (ANDAR1 .GT. .46452) THEN
      POWER(11) = POWER(11) + 1
ENDIF
IF (ANDAR1 .GT. .51170) THEN
      POWER(12) = POWER(12) + 1
ENDIF
IF (ANDAR1 .GT. .58377) THEN
      POWER(13) = POWER(13) + 1
ENDIF
IF (ANDAR1 .GT. .68950) THEN
      POWER(14) = POWER(14) + 1
ENDIF
```

```
            IF (ANDAR1 .GT. .90866) THEN
                POWER(15) = POWER(15) + 1
            ENDIF
            IF (ANDAR2 .GT. .44203) THEN
                POWER(16) = POWER(16) + 1
            ENDIF
            IF (ANDAR2 .GT. .50275) THEN
                POWER(17) = POWER(17) + 1
            ENDIF
            IF (ANDAR2 .GT. .57780) THEN
                POWER(18) = POWER(18) + 1
            ENDIF
            IF (ANDAR2 .GT. .71245) THEN
                POWER(19) = POWER(19) + 1
            ENDIF
            IF (ANDAR2 .GT. 1.05927) THEN
                POWER(20) = POWER(20) + 1
            ENDIF
            IF (CRVM .GT. .087204) THEN
                POWER(22) = POWER(22) + 1
            ENDIF
            IF (CRVM .GT. .078210) THEN
                POWER(21) = POWER(21) + 1
            ENDIF
            IF (CRVM .GT. .100425) THEN
                POWER(23) = POWER(23) + 1
            ENDIF
            IF (CRVM .GT. .120583) THEN
                POWER(24) = POWER(24) + 1
            ENDIF
            IF (CRVM .GT. .170314) THEN
                POWER(25) = POWER(25) + 1
            ENDIF
            IF(CRVM2 .GT. .071429) THEN
                POWER(26) = POWER(26) + 1
            ENDIF
            IF (CRVM2 .GT. .081378) THEN
                POWER(27) = POWER(27) + 1
            ENDIF
            IF (CRVM2.GT. .096362) THEN
                POWER(28) = POWER(28) + 1
            ENDIF
            IF (CRVM2 .GT. .124018) THEN
                POWER(29) = POWER(29) + 1
            ENDIF
            IF (CRVM2 .GT. .181580) THEN
                POWER(30) = POWER(30) + 1
            ENDIF
  100   CONTINUE
        PRINT '(A)', '1'
        PRINT *
        PRINT *
```

91

```
      PRINT *,'AGAINST THE CAUCHY DISTRIBUTION'
      PRINT *,'THE REJECTIONS AT N = ',NR,' ARE AS FOLLOWS:'
      PRINT *
      PRINT '(T7,A,T17,A,T27,A,T37,A,T47,A)','.80','.85'
     +      '.90','.95','.99'
      PRINT *, 'FOR LILIEFORS: '
      PRINT '(T6,I4,T16,I4,T26,I4,T36,I4,T46,I4)',
     +      (POWER(I),I=1,5)
      PRINT *, 'FOR LILIEFORS DOUBLED: '
      PRINT '(T6,I4,T16,I4,T26,I4,T36,I4,T46,I4)',
     +      (POWER(I),I=6,10)
      PRINT *, 'FOR ANDERSON-DARLING: ',
      PRINT '(T6,I4,T16,I4,T26,I4,T36,I4,T46,I4)',
     +      (POWER(I),I=11,15)
      PRINT *, 'FOR ANDERSON-DARLING DOUBLED: '
      PRINT '(T6,I4,T16,I4,T26,I4,T36,I4,T46,I4)',
     +      (POWER(I),I=16,20)
      PRINT *, 'FOR CRAMER-VON MISES: '
      PRINT '(T6,I4,T16,I4,T26,I4,T36,I4,T46,I4)',
     +      (POWER(I),I=21,25)
      PRINT *, 'FOR CRAMER-VON MISES DOUBLED: '
      PRINT '(T6,I4,T16,I4,T26,I4,T36,I4,T46,I4)',
     +      (POWER(I),I=26,30)
      PRINT *
      PRINT *
      PRINT *
      DO  500 I = 1,30
          PWR(I) = POWER(I)/5000.0
500   CONTINUE
      PRINT *, 'THE POWERS AT N = ',NR,' ARE AS FOLLOWS: '
      PRINT *, 'FOR LILIEFORS: '
      PRINT '(T5,F6.4,T15,F6.4,T25,F6.4,T35,F6.4,T45,F6.4)',
     +      (PWR(I),I=1,5)
      PRINT *, 'FOR LILIEFORS DOUBLED: '
      PRINT '(T5,F6.4,T15,F6.4,T25,F6.4,T35,F6.4,T45,F6.4)',
     +      (PWR(I),I=6,10)
      PRINT *, 'FOR ANDERSON-DARLING: '
      PRINT '(T5,F6.4,T15,F6.4,T25,F6.4,T35,F6.4,T45,F6.4)',
     +      (PWR(I),I=11,15)
      PRINT *, 'FOR ANDERSON-DARLING DOUBLED: '
      PRINT '(T5,F6.4,T15,F6.4,T25,F6.4,T35,F6.4,T45,F6.4)',
     +      (PWR(I),I=16,20)
      PRINT *, 'FOR CRAMER-VON MISES: '
      PRINT '(T5,F6.4,T15,F6.4, T25,F6.4,T35,F6.4,T45,F6.4)',
     +      (PWR(I),I=21,25)
      PRINT *, 'FOR CRAMER-VON MISES DOUBLED: '
      PRINT '(T5,F6.4,T15,F6.4,T25,F6.4,T35,F6.4,T45,F6.4)',
     +      (PWR(I),I=26,30)
      PRINT *
      PRINT *
```

```
PRINT *, 'COUNT1 = ',COUNT(1)
PRINT *, 'COUNT2 = ',COUNT(2)
PRINT *, 'COUNT3 = ',COUNT(3)
PRINT *, 'COUNT4 = ',COUNT(4)
END
```

Vita

Thomas John Ream was born on 17 November 1950, to Vincent and Margaret Ream in Welch, West Virginia. He received his entire elementary and secondary education in Morgantown, West Virginia, where he graduated from Morgantown High School in 1968. He attended West Virginia University for one year before entering the United States Air Force Academy in June 1969. After graduation from the Academy as a History major with a Bachelor of Science degree in June 1973, Captain Ream attended pilot training at Laughlin AFB, near Del Rio, Texas--earning his pilot's wings in November 1974. Captain Ream was subsequently assigned to the 15th Military Airlift Squadron (MAC) at Norton AFB, California. Captain Ream served there as a C-141A co-pilot, Aircraft Commander, and Instructor Aircraft Commander until May 1980, when he was assigned to study Operations Research at the School of Engineering, Air Force Institute of Technology.

Permanent Address: 1105 Windsor Ave.
                     Morgantown, W.Va. 26505

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFIT/GOR/MA/81D-9	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A NEW GOODNESS OF FIT TEST FOR NORMALITY WITH MEAN AND VARIANCE UNKNOWN		5. TYPE OF REPORT & PERIOD COVERED Master's Thesis
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Thomas J. Ream Captain USAF		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Air Force Institute of Technology(AFIT/EN) Wright-Patterson AFB, OH 45433		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Institute of Technology(AFIT/EN) Wright-Patterson AFB, OH 45433		12. REPORT DATE December 1981
		13. NUMBER OF PAGES 105
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

1 5 APR 1982

18. SUPPLEMENTARY NOTES

Approved for public release; IAW AFR 190-17

Lynn E. Wolaver

Dean for Research and
Professional Development
Air Force Institute of Technology (ATC)
Wright-Patterson AFB, OH   45433

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Normal distribution
Sample reflection
Anderson-Darling
Power study

Cramer-von Mises
Kolmogorov-Smirnov
New statistic

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A new technique for calculating known goodness of fit statistics for the Normal distribution is investigated. Samples are generated for a Normal (0,1) distribution. The means of these samples are calculated and the samples are doubled by reflecting sample data points about the individual sample means. This reflection of data points about the mean is a new technique for

Block 20.

generating modified statistics. After the sample is doubled,
critical values are calculated for these modified Kolmogorov-
Smirnov, Anderson-Darling, and Cramer-von Mises statistics.
Critical values are for the original sample sizes. An extensive
power study is done to test the power of the three new statis-
tics' critical values versus the power for the same three statis-
tics, calculated without reflection.

Powers of the new statistics are symptotically slightly higher
than the powers of their non-reflected counterparts, when the
distribution tested is also symmetrical. The powers of new
statistics are substantially lower when the distribution tested
is non-symmetrical. The powers are substantially higher for the
modified statistics when the continuous Uniform distribution is
tested.

Complete tables of critical values for sample sizes n =3
through n = 60 are included for the modified Kolmogorov-
Smirnov and Anderson-Darling statistics.